PH113: Physics (B.Tech. I)

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Section L, O & P

PH113: Physics (B.Tech. I)

Syllabus

Module-1: Physical Optics
Polarization, Malus' Law, Brewster's Law, Double Refraction, Interference in thin films (Parallel films), Interference in wedge-shaped layers, Newton's rings, Fraunhofer diffraction by single slit, Double slit. Elementary ideas of fibre optics and application of fibre optic cables. [8]

Module-2: Electromagnetic Theory
Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

Module-3: Special Theory of Relativity
Introduction, Inertial frame of reference, Galilean transformations, Postulates, Lorentz transformations and its conclusions, Length contraction, time dilation, velocity addition, Mass change, Einstein's mass energy relation. [6]

Module-4: Quantum Mechanics
Planck's theory of black-body radiation, Compton effect, Wave particle duality, De Broglie waves, Davisson and Germer's experiment, Uncertainty principle, Brief idea of Wave Packet, Wave Function and its physical interpretation, Schrodinger equation in one-dimension, free particle, particle in an infinite square well. [9]

Module-5: Modern Physics
Laser – Spontaneous and stimulated emission, Einstein's A and B coefficients, Population inversion, Light amplification, Basic laser action, Ruby and He-Ne lasers, Properties and applications of laser radiation, Nuclear Physics- Binding Energy Curve, Nuclear Force, Liquid drop model, Introduction to Shell model, Applications of Nuclear Physics, Concept of Plasma Physics, and its applications. [9]

Text books:
1: A. Ghatak, Optics, 4th Edition, Tata Mcgraw Hill, 2009
2: Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press (2001)
3: Arthur Beiser, Concept of Modern Physics, 6th edition 2009, Tata McGraw- Hill
4. F. F. Chen, Introduction to Plasma Physics and controlled Fusion, Springer, Edition 2016.

Reference books: 1: Fundamentals of Physics, Halliday, Walker and Resnick

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

Text Book: Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press (Seventh Edition, 2018)

Reference Book: David J. Griffiths, Introduction to Electrodynamics, Pearson (Fourth Edition, 2014)

Class structure: 4 Lectures including 1 Tutorial per week. (8 hours ~ 2 weeks for this module!)

Module 2

Date: 26.09.2023

Lecture: 1

Module 2 (Electromagnetic Theory)

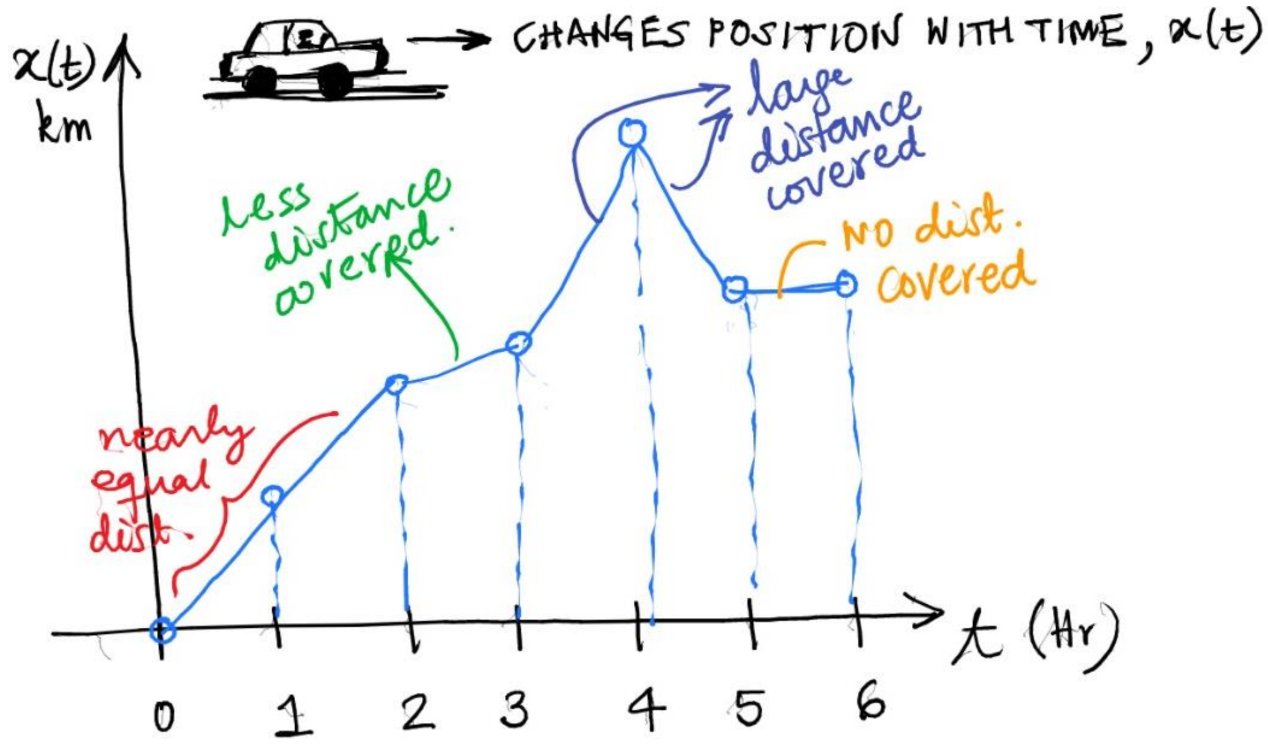
Introduction

- Mathematical preliminaries
  - Gradient, Divergence, Curl and their applications
  - Gauss theorem
  - Stokes theorem

Module 2 (Electromagnetic Theory)

Ultrashort Introduction to Calculus

- A fundamental question in classical physics:
- A car/particle/object is moving (= changing its position with time).
- Can we predict where the car/particle/object will be in some time instance in future?



- X-axis: time (equal intervals)
- Y-axis: position of the car
- X(t) is a **function**.

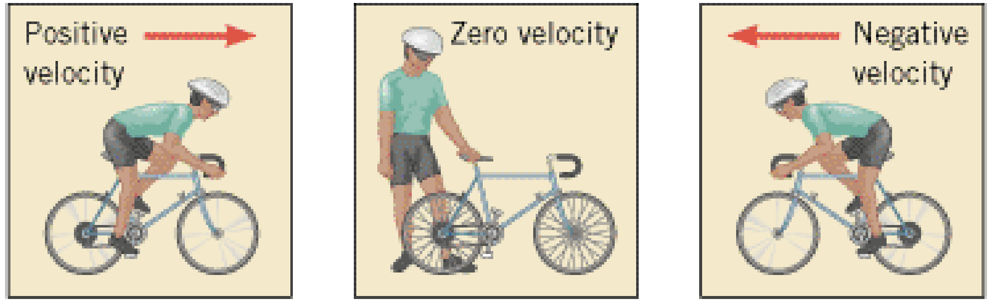
- General scenario: it covers different distance in the same time intervals.
- Natural guess (almost all of you): it SPEEDS UP or SLOWS DOWN at different times..

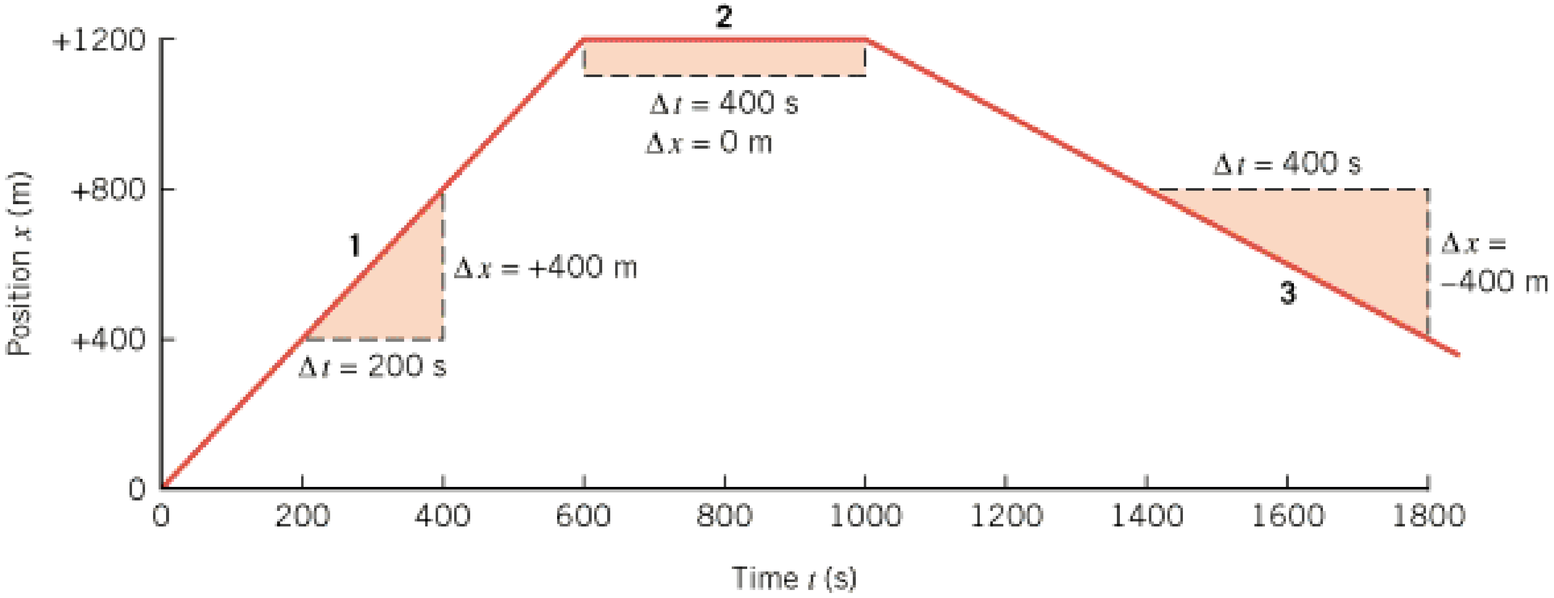
- Quantifiable information can be obtained!!

Module 2 (Electromagnetic Theory)

Introduction

❑ Ultrashort intro to Calculus





- If equal distance covered in equal intervals of time: velocity, v= constant, and acceleration, a = 0.

- Positive a ?
- Negative a ?

Also take a look at:

https://www.engineersedge.com/physics/acceleration-animation.htm

Module 2 (Electromagnetic Theory)

Introduction

❑ Velocity and Acceleration

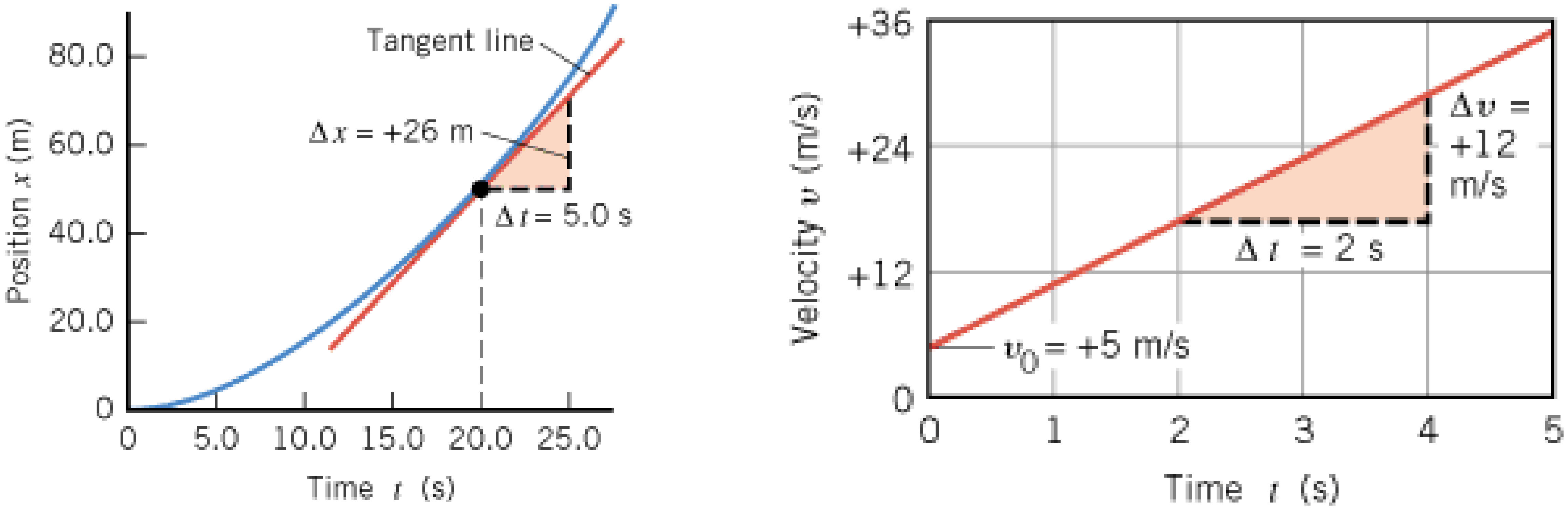
- $\Delta x(t)$ = distance covered in the time interval Δt.
- $v(t) = \Delta x(t)/\Delta t$ = "rate of change in position" = velocity/speed. This informs us about how fast or slow the car is moving. v is a function of time, $v(t)$.
- Tangent at that time.
- $\mathrm{a(t)} = \Delta v(t)/\Delta t$ = "rate of change in velocity" = acceleration. This informs us about how quickly the drive is changing his/her speed. Also $\mathbf{a}$ is function of time, $\mathbf{a(t)}$.
- Tangent to the velocity profile at that time.
- In the limit Δt --> 0, a better approximation to the function is generally possible – the notion of **derivative.**

- Lim Δt --> 0 : Δx/Δt --> del x(t)/ del t = v(t)
- Δx/Δt --> del v(t)/ del t = a(t)

Module 2 (Electromagnetic Theory)

Introduction

❑ Velocity and Acceleration



Module 2 (Electromagnetic Theory)

Introduction

❑ Derivatives of some standard functions

1. Constant Rule $\longrightarrow \frac{d}{dx}(n) = 0$
2. Constant Multiple Rule $\longrightarrow \frac{d}{dx}[nf(x)] = nf'(x)$
3. Sum Rule $\longrightarrow \frac{d}{dx}[a(x) + b(x)] = a'(x) + b'(x)$
4. Difference Rule $\longrightarrow \frac{d}{dx}[a(x) - b(x)] = a'(x) - b'(x)$
5. Product Rule $\longrightarrow \frac{d}{dx}[a(x)b(x)] = a(x)b'(x) + b(x)a'(x)$
6. Quotient Rule $\longrightarrow \frac{d}{dx}[\frac{a(x)}{b(x)}] = \frac{b(x)a'(x) - a(x)b'(x)}{[b(x)]^2}$
7. Power Rule $\longrightarrow \frac{d}{dx}(x^n) = nx^{n-1}$
8. Chain Rule $\longrightarrow \frac{d}{dx}f(a(x)) = f'(a(x))a'(x)$

Module 2 (Electromagnetic Theory)

Introduction

- Derivatives of some standard functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Many more functions, look 'em up …

Module 2 (Electromagnetic Theory)

Introduction

- Higher Order Derivatives

- a(t) $= \frac{\Delta v(t)}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$
- a is a **second order derivative** of $x(t)$.

Module 2 (Electromagnetic Theory)

Introduction

❑ Scalar and Vector Fields

- ❑ Consider the temperature distribution in the room: $T(x, y, z)$
  - ❑ At each point in the room, Temperature (T) is known.
  - ❑ This is an example of a **scalar field** since T is a scalar quantity.

- ❑ Similarly, consider the (average) flow of air particles in the room quantified by the (average) velocity of air particles : $\boldsymbol{v}(x, y, z)$
  - ❑ At each point in the room, the velocity ($\boldsymbol{v}$) is known.
  - ❑ This is an example of a **vector field** since $\boldsymbol{v}$ is a vector quantity.

- ❑ To compute the derivatives/integral of a vector field, one needs to do this for each component
- ❑ For a function of multiple variable, we need **multivariate calculus**.

Module 2 (Electromagnetic Theory)

Introduction

- Ordinary derivatives v/s Partial derivatives

Ordinary derivatives	Partial derivatives
- Concerns single-variable functions.	- Applied to multivariable functions.
- Measures the rate of change with respect to one variable.	- Measures the rate of change with respect to one variable while holding others constant.
- Denoted as dy/dx.	- Denoted as ∂f/∂x, ∂f/∂y, etc.

-The swirly-d (symbol: ∂, called "del") is used to distinguish partial derivatives from ordinary single-variable derivatives.

$$\frac{\partial f}{\partial x} = \underbrace{\frac{\partial}{\partial x} x^2 y}_{\substack{\text{Treat } y \text{ as constant;} \\ \text{take derivative.}}} = 2xy$$

Treat the "other" variable as a "constant" !

$$\frac{\partial f}{\partial y} = \underbrace{\frac{\partial}{\partial y} x^2 y}_{\substack{\text{Treat } x \text{ as constant;} \\ \text{take derivative.}}} = x^2 \cdot 1$$

-Total derivative of a function: $df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz$

Module 2 (Electromagnetic Theory)

Introduction

- **Gradient (grad):**
- Definition: The gradient is a vector operator that represents the rate of change of a scalar field.
- Mathematical Expression:

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial f}{\partial z}\right)\boldsymbol{k}$$

- Operates on a scalar field, produces a vector function.
- Interpretation: The gradient points in the direction of the steepest increase of the scalar field. In temperature mapping, $T(x, y, z)$, the gradient of temperature indicates the direction of maximum temperature increase.
- Useful in optimization/ML/AI problem: the widely used gradient descent algorithm.

Module 2 (Electromagnetic Theory)

Introduction

- Gradient (grad):

- Consider the temperature of a room $T(x, y, z)$. Variation in temperature is given by:

- $dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$

$$= \left[\left(\frac{\partial T}{\partial x}\right) \boldsymbol{i} + \left(\frac{\partial T}{\partial y}\right) \boldsymbol{j} + \left(\frac{\partial T}{\partial z}\right) \boldsymbol{k}\right] . [dx\ \boldsymbol{i} + dy\ \boldsymbol{j} + dz\ \boldsymbol{k}]$$

$$= \boldsymbol{\nabla} T \,.\, d\boldsymbol{l}$$

Where, $\boldsymbol{\nabla} T \equiv \left(\frac{\partial T}{\partial x}\right) \boldsymbol{i} + \left(\frac{\partial T}{\partial y}\right) \boldsymbol{j} + \left(\frac{\partial T}{\partial z}\right) \boldsymbol{k}$ [i.e., $\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}\right) \boldsymbol{i} + \left(\frac{\partial}{\partial y}\right) \boldsymbol{j} + \left(\frac{\partial}{\partial z}\right) \boldsymbol{k}$]

and the line element $d\boldsymbol{l} = dx\ \boldsymbol{i} + dy\ \boldsymbol{j} + dz\ \boldsymbol{k}$.

Module 2 (Electromagnetic Theory)

Introduction

- **Gradient (grad):**
- Variation in temperature $T(x, y, z)$ of a room is given by: $dT = \boldsymbol{\nabla} T \,.\, d\boldsymbol{l}$
- Geometrical interpretation: $dT = \boldsymbol{\nabla} T \,.\, d\boldsymbol{l} = |\boldsymbol{\nabla} T|\, |d\boldsymbol{l}|\, cos\theta$
- If $|d\boldsymbol{l}|$ is fixed, then maximum variation in $T(x, y, z)$ is when $\theta = 0$ (i.e., $cos\theta = 1$)
- So, dT is maximum along the direction of $\boldsymbol{\nabla} T$.
- Under this condition (i.e., $\theta = 0$), $\boldsymbol{\nabla} T = \left|\frac{dT}{d\boldsymbol{l}}\right|$
- Suppose you are on a hilltop. Look for the direction of steepest ascent. It gives the direction of the gradient. The value of the slope along this direction gives the magnitude of the gradient.

Module 2 (Electromagnetic Theory)

Introduction

- Gradient Identities:

a. $\nabla(f+g) = \nabla f + \nabla g$

b. $\nabla(cf) = c\nabla f$, *for any* ***constant*** c

c. $\nabla(fg) = (\nabla f)g + f(\nabla g)$

d. $\nabla(f/g) = \left(g\nabla f - f\nabla g\right)/g^2$ *at points* $\mathbf{x}$ *where* $g(\mathbf{x}) \neq 0$.

e. $\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) - (\nabla\times\mathbf{F})\times\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + (\mathbf{F}\cdot\nabla)\mathbf{G}$

where $$(\mathbf{G}\cdot\nabla)\mathbf{F} = \mathbf{G}_1\frac{\partial\mathbf{F}}{\partial x} + \mathbf{G}_2\frac{\partial\mathbf{F}}{\partial y} + \mathbf{G}_3\frac{\partial\mathbf{F}}{\partial z}$$

Examples:

1. For $f(x,y) = x + 3y^2$, $\nabla f = \boldsymbol{i} + 6y\,\boldsymbol{j}$
2. For $f(x,y) = \sin(x)\,e^y$, $\nabla f = \cos(x)\,e^y\,\boldsymbol{i} + \sin(x)\,e^y\,\boldsymbol{j}$

Module 2 (Electromagnetic Theory)

Introduction

- Gradient (grad):

- **Solve**: The position of a point is given by $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$. Find $\boldsymbol{\nabla}$r.

- Note: Here $r = |\boldsymbol{r}| = \sqrt{(x^2 + y^2 + z^2)}$ and $\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial}{\partial z}\right)\boldsymbol{k}$
- Hence $\boldsymbol{\nabla} r = \left(\frac{\partial r}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial r}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial r}{\partial z}\right)\boldsymbol{k}$
- Now, $\left(\frac{\partial r}{\partial x}\right) = \frac{x}{\sqrt{(x^2+y^2+z^2)}} = \frac{x}{r}$.
- Similarly, $\left(\frac{\partial r}{\partial y}\right) = \frac{y}{r}$ and $\left(\frac{\partial r}{\partial z}\right) = \frac{z}{r}$
- Finally, we get $\boldsymbol{\nabla} r = \left(\frac{x}{r}\right)\boldsymbol{i} + \left(\frac{y}{r}\right)\boldsymbol{j} + \left(\frac{z}{r}\right)\boldsymbol{k} = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}}$

Module 2 (Electromagnetic Theory)

Introduction

- Divergence (div):

- Definition: The divergence is a scalar operator that measures the spread or dispersion of a vector field from a point.

- Mathematical Expression:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial F_x}{\partial x}\right) + \left(\frac{\partial F_y}{\partial y}\right) + \left(\frac{\partial F_z}{\partial z}\right)$$

- Operates (dot product) on a vector, produces a scalar function. (Dot product for Divergence!)

- **Interpretation**: Rate of outward/inward flow of the vector field (flux through a surface) at the point where divergence is evaluated. Positive divergence indicates a source (outward flow), while negative divergence indicates a sink (inward flow).

Module 2 (Electromagnetic Theory)

Introduction

❑ Divergence Identities:

a. $\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}$

b. $\nabla \cdot (c\mathbf{F}) = c\nabla \cdot \mathbf{F}$, *for any* ***constant*** *c*

c. $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}$

d. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

Examples:

1. For $\boldsymbol{F}(x, y) = 6x^2\,\boldsymbol{i} + 4y\,\boldsymbol{j}$, $div(\boldsymbol{F}) = 12x + 4$

2. For $\boldsymbol{F}(x, y, z) = x^2\,\boldsymbol{i} + 2z\,\boldsymbol{j} - y\,\boldsymbol{k}$, $div(\boldsymbol{F}) = 2x$

Module 2 (Electromagnetic Theory)

Introduction

- **Curl:**

- Definition: The curl is a vector operator that measures the rotation or circulation of a vector field at a point.

- Mathematical Expression:

$$\nabla \times \mathbf{F} = \left[\left(\frac{\partial F_z}{\partial y}\right) - \left(\frac{\partial F_y}{\partial z}\right)\right] \boldsymbol{i} + \left[\left(\frac{\partial F_x}{\partial z}\right) - \left(\frac{\partial F_z}{\partial x}\right)\right] \boldsymbol{j} + \left[\left(\frac{\partial F_y}{\partial x}\right) - \left(\frac{\partial F_x}{\partial y}\right)\right] \boldsymbol{k}$$

- Operates on a vector field, produces a vector field. (Cross product for Curl!)

- **Interpretation**: Determines the circulation of a vector field. The curl vector points in the direction of the axis of rotation and its magnitude represents the strength of rotation.

Module 2 (Electromagnetic Theory)

Introduction

❑ Curl Identities:

a. $\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}$

b. $\nabla \times (c\mathbf{F}) = c\nabla \times \mathbf{F}$, *for any* ***constant*** *c*

c. $\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f\nabla \times \mathbf{F}$

d. $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - (\nabla \cdot \mathbf{F})\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

Examples:

1. For $\boldsymbol{F}(x, y, z) = y^3\, \boldsymbol{i} + xy\, \boldsymbol{j} - z\, \boldsymbol{k}$, $curl(\boldsymbol{F}) = (y - 3y^2)\, \boldsymbol{k}$
2. For $\boldsymbol{F}(x, y, z) = x\, \boldsymbol{i} + y\, \boldsymbol{j} + z\, \boldsymbol{k} = \boldsymbol{r}$, $curl(\boldsymbol{r}) = 0$
3. For $\boldsymbol{v} = -y\, \boldsymbol{i} + x\, \boldsymbol{j}$, $curl(\boldsymbol{v}) = 2\, \boldsymbol{k}$
4. For $\boldsymbol{v} = x\, \boldsymbol{j}$, $curl(\boldsymbol{v}) = \boldsymbol{k}$

Module 2 (Electromagnetic Theory)

Introduction

- Laplacian:

$$\nabla^2 f = \boldsymbol{\nabla} . (\boldsymbol{\nabla} f) = \left[\left(\frac{\partial}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial}{\partial z}\right)\boldsymbol{k}\right] . \left[\left(\frac{\partial f}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial f}{\partial z}\right)\boldsymbol{k}\right]$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial^2 f}{\partial z^2}\right)$$

- Identities:

a. $\boldsymbol{\nabla}^2(f+g) = \boldsymbol{\nabla}^2 f + \boldsymbol{\nabla}^2 g$

b. $\boldsymbol{\nabla}^2(cf) = c\boldsymbol{\nabla}^2 f$, *for any* ***constant*** c

c. $\boldsymbol{\nabla}^2(fg) = f\boldsymbol{\nabla}^2 g + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + g\boldsymbol{\nabla}^2 f$

- Operates on a scalar field, produces a scalar function.

Module 2 (Electromagnetic Theory)

Introduction

- **Laplacian:**

$$\nabla^2 f = \nabla . (\nabla f) = \left[\left(\frac{\partial}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial}{\partial z}\right)\boldsymbol{k}\right] . \left[\left(\frac{\partial f}{\partial x}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)\boldsymbol{j} + \left(\frac{\partial f}{\partial z}\right)\boldsymbol{k}\right] = \left(\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial^2 f}{\partial z^2}\right)$$

- **Example**: $f(x, y, z) = x^2 + y^2 + z^2$, calculate $\nabla^2 f$.

  - Step by step: $\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = 2z$
  - $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2$. Similarly, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =2$
  - Therefore, $\nabla^2 f == \left(\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial^2 f}{\partial z^2}\right) = 2 + 2 + 2 = 6.$

Module 2 (Electromagnetic Theory)

Introduction

- DCG:

a. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ (divergence of curl)

b. $\nabla \times (\nabla f) = 0$ (curl of gradient)

c. $\nabla \cdot (f\{\nabla g \times \nabla h\}) = \nabla f \cdot (\nabla g \times \nabla h)$

d. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

e. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$ (curl of curl)

Curl of a radial outward/inward vector is always zero!

(will be useful in talking about conservative nature of $\boldsymbol{E}$)

Module 2 (Electromagnetic Theory)

Introduction

❑Fundamental theorem of calculus:

Given *f* is

- continuous on interval [a, b]
- *F* is any function that satisfies *F'*(x) = *f*(x)

Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

❑Integration as area under the curve:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Module 2 (Electromagnetic Theory)

Introduction

❑Integration: Inverse of Derivative

- $\int 1\, dx = x + C$
- $\int a\, dx = ax + C$
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C;\ n \neq -1$
- $\int \sin x\, dx = -\cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \csc^2 x\, dx = -\cot x + C$
- $\int \sec x(\tan x)\, dx = \sec x + C$
- $\int \csc x(\cot x)\, dx = -\csc x + C$
- $\int \frac{1}{x}\, dx = \ln|x| + C$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = \frac{a^x}{\ln a} + C;\ a > 0, a \neq 1$
- $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \sec^{-1} x + C$
- $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
- $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
- $\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) + \int \tan^{n-2}(x)dx$
- $\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
- $\int \csc^n(x)dx = \frac{-1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)dx$

Module 2 (Electromagnetic Theory)

Introduction

- Vector Integration:
- Extends the concept of integration to vector-valued functions.
- Involves integrating vector quantities, such as displacement, force, or velocity.
- Essential in various fields, including physics, engineering, and mathematics.
- Notations:
    - One dimensional integral between x_0, x_1
    - One dimensional integral over a closed loop: $\oint$ (integral sign with a circle)
    - Surface or volume integrals: $\iint$ (double integral sign) or $\int dxdy$, or
      $\iiint$ (double integral sign) or $\int dxdydz$

Module 2 (Electromagnetic Theory)

Introduction

❑ Vector Integration:

❑ **Example**:

1. Area of a rectangle of sides $\boldsymbol{a}, \boldsymbol{b}$:

$\int_0^a \int_0^b dx\, dy = \int_0^a dx\ \int_0^b dy = ab$ (Note: separation possible in this case, not so in general!)

2. Area of a circle: circle is defined as $\boldsymbol{x^2 + y^2 = r^2}$.

Note: only **one** effective variable (the other depends on it), separation of variables is not possible.

Try solving it

Module 2 (Electromagnetic Theory)

Introduction

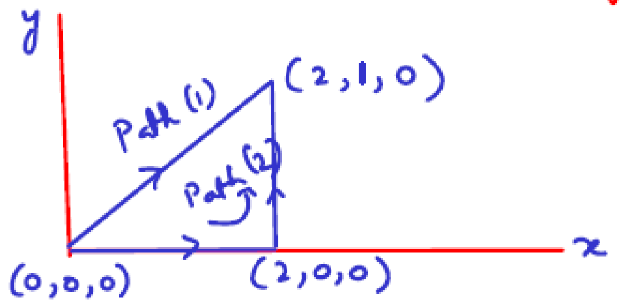
❑ Fundamental theorem of Gradients

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).$$

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

Q. Let $f = xy^2$, $\bar{a} \equiv (0,0,0)$ and $\bar{b} \equiv (2,1,0)$.
Check the fundamental theorem of gradients using:



← Check this

Module 2 (Electromagnetic Theory)

Introduction

❑ Gauss theorem (Divergence theorem) and its physical meaning

❑ $\int_V (\boldsymbol{\nabla}.\boldsymbol{v})\, d\tau = \oint_S \boldsymbol{v}.d\boldsymbol{a}$

❑ Relates volume to surface integral (bulk-boundary correspondence!)

❑ Physical Meaning:

- Gauss's Theorem relates the flux of a vector field $\boldsymbol{v}$ through a closed surface S to the divergence of $\boldsymbol{v}$ within the enclosed volume V.
- It tells us that the total flux leaving or entering a closed surface is equal to the net source or sink of the vector field inside the volume.

❑ Applications:

- **Electric Flux**: In electromagnetism, Gauss's theorem helps calculate electric flux through a closed surface due to charges within a volume, providing insights into electric fields and charge distributions.

Module 2 (Electromagnetic Theory)

Introduction

- Stokes theorem and its physical meaning

- $\int_S (\boldsymbol{\nabla} \times \boldsymbol{v}) . d\boldsymbol{a} = \oint_C \boldsymbol{v} . d\boldsymbol{l}$
- Relates surface integrals to line integrals (bulk boundary correspondence, again!)
- Physical Meaning:
  - Stokes' Theorem connects the circulation of a vector field $\boldsymbol{v}$ around a closed curve C to the curl of $\boldsymbol{v}$ over the surface S that the curve bounds.
  - It helps us understand how circulation around a curve is related to the rotation of the vector field over the surface.
- Applications:
  - **Fluid Dynamics**: Stokes' theorem is fundamental in fluid dynamics, where it relates the circulation of velocity around a closed path to the vorticity within the enclosed region, helping analyze fluid flow patterns.

Module 2 (Electromagnetic Theory)

Introduction

- Show that
  - $\nabla . (\nabla \times \boldsymbol{v}) = 0$
  - $\nabla \times (\nabla f) = 0$

Next Class

Questions?

Module 2

Date: 27.09.2023

Lecture: 2

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

Text Book: Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press (Seventh Edition, 2018)

Reference Book: David J. Griffiths, Introduction to Electrodynamics, Pearson (Fourth Edition, 2014)

Class structure: 4 Lectures including 1 Tutorial per week. (8 hours ~ 2 weeks for this module!)

Module 2 (Electromagnetic Theory)

Electrostatics

- Coulomb's law

- Force on a test charge Q due to a single point charge q, that is at rest a distance $\mathscr{r}$ away is given by Coulomb's law:

- $\bar{F} = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{\mathscr{r}^2} \hat{\mathscr{r}}$ [ϵ_0 is the permittivity in free space] $\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N} \cdot \mathrm{m}^2}$

- Force is proportional to the product of the *charges* and inversely proportional to the square of the *separation distance*.

- Note: $\mathscr{r}$ is the separation **vector** from r' (the location of q) to r (the location of Q): $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$

- The force points along the line from q to Q; it is **repulsive** if q and Q have the same sign, and **attractive** if their signs are opposite.

Module 2 (Electromagnetic Theory)

Electrostatics

❑ The Electric Field

❑ Consider several point charges $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1$, $\mathscr{r}_2$, …., $\mathscr{r}_n$ from Q, the total force on Q is given by the superposition principle:

$$\bar{F} = \bar{F_1} + \bar{F_2} + \ldots\ldots. = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\,\hat{r_1} + \frac{q_2}{r_2^2}\,\hat{r_2} + \ldots..\right) = Q\bar{E}$$

Where $\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^{\,2}}\,\hat{\mathscr{r}}_i$ (discrete charge distribution)

❑ $\bar{E}$ is called the **electric field** of the source charges.

❑ Note: $\bar{E}$ is a function of position (**r**), because the separation vectors $\mathscr{r}_i$ depend on the location of the field point. But it makes no reference to the test charge Q.

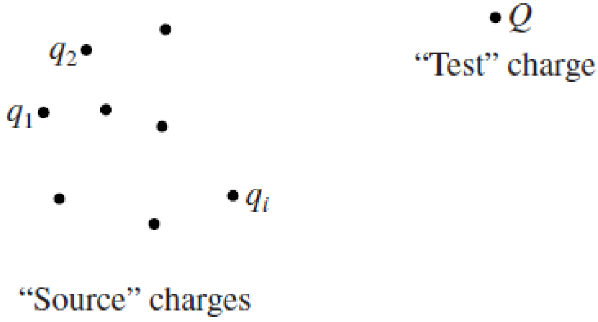


FIGURE 1

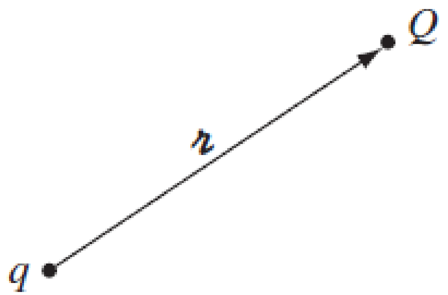
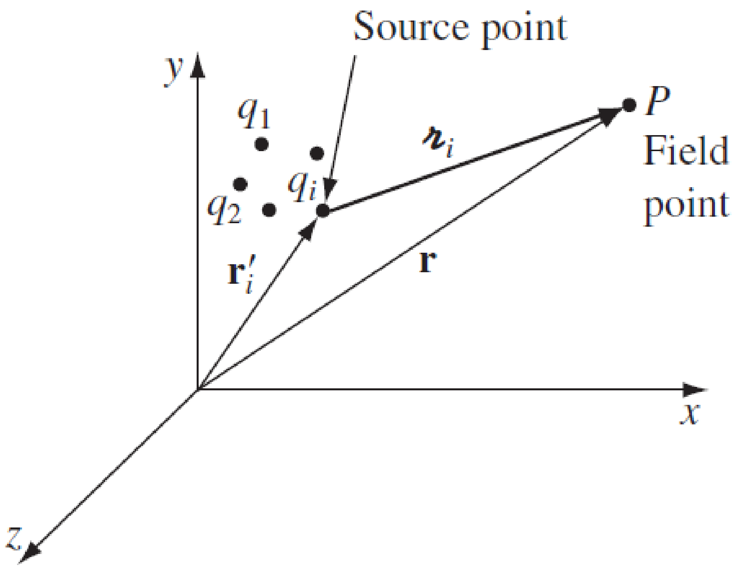


FIGURE 2

Module 2 (Electromagnetic Theory)

Electrostatics



- The Electric Field

- Discrete charge distribution: $\bar{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2}\, \hat{\boldsymbol{\mathscr{r}}}_i$

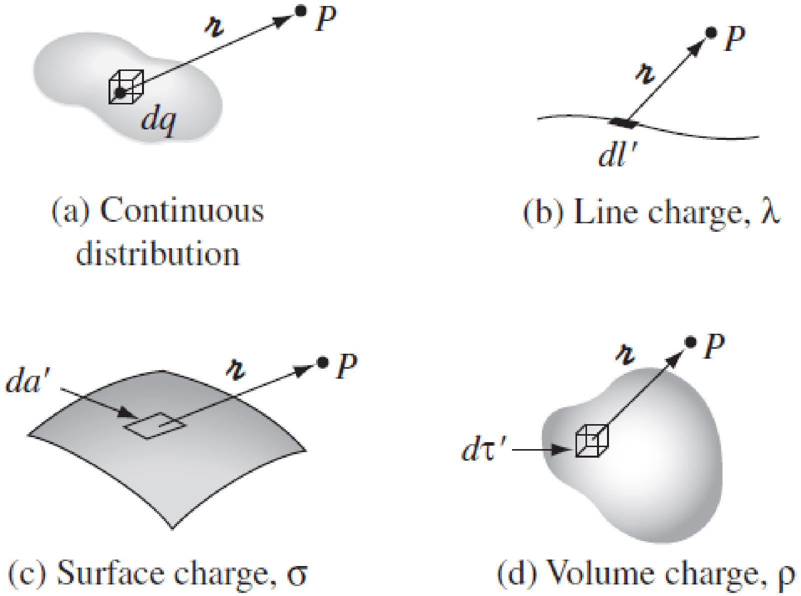
- $\bar{E}$ is called the **electric field** of the source charges. It is a **vector** quantity that varies from point to point and is determined by the configuration of source charges.

- Physically, $\boldsymbol{E(r)}$ is the force per unit charge that would be exerted on a test charge, if you were to place one at P.

Module 2 (Electromagnetic Theory)

Electrostatics

- ❑ The Electric Field



(a) Continuous distribution
(b) Line charge, λ
(c) Surface charge, σ
(d) Volume charge, ρ

- ❑ Continuous Charge Distributions: Instead of assuming that the source of the field is a collection of discrete point charges q_i, if the charge is distributed continuously over some region, the sum becomes an integral: $\bar{\boldsymbol{E}}(\bar{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}$

- ❑ If the charge is spread out along a **line** (Fig. b), then $dq = \lambda dl'$ (λ is the charge-per-unit-length and dl' is an element of length along the line).

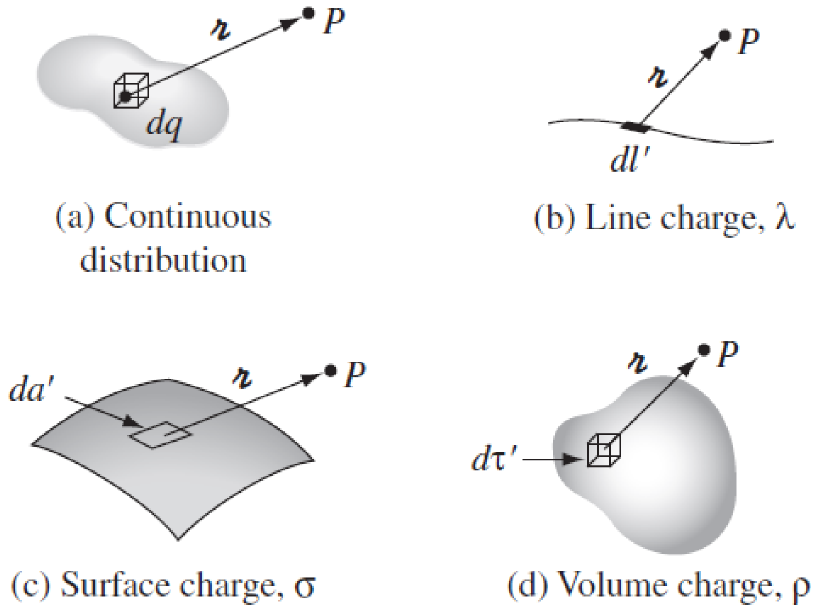
- ❑ If the charge is smeared out over a **surface** (Fig. c), then $dq = \sigma da'$ (σ is the charge-per-unit-area and da' is an element of area on the surface).

- ❑ If the charge fills a **volume** (Fig. d), then $dq = \rho d\tau'$ (ρ is the charge-per-unit-volume and $d\tau'$ is an element of volume).

Module 2 (Electromagnetic Theory)

Electrostatics

❑The Electric Field



❑For a continuous charge distribution: $\bar{\boldsymbol{E}}(\bar{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}}$

❑The electric field of a line charge is: $\bar{\boldsymbol{E}}(\bar{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda\, dl'}{\mathscr{r}^2} \hat{\mathscr{r}}$

❑For a surface charge: $\bar{\boldsymbol{E}}(\bar{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma\, da'}{\mathscr{r}^2} \hat{\mathscr{r}}$

❑For a volume charge: $\bar{\boldsymbol{E}}(\bar{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho\, d\tau'}{\mathscr{r}^2} \hat{\mathscr{r}}$ (often referred to as the Coulomb law)

Module 2 (Electromagnetic Theory)

Electrostatics

- The Electric Flux

- Total flux of $\boldsymbol{E}$ through a surface *S*: $\Phi_E \equiv \int_S \boldsymbol{E}.\boldsymbol{da}$

- This is a measure of the "*number of field lines*" passing **normally** through the surface *S*.

- $\boldsymbol{E}.\boldsymbol{da}$ is proportional to the number of lines passing through the infinitesimal area $\boldsymbol{da}$. (It is the area in the plane perpendicular to $\boldsymbol{E}$).

- For a closed surface, the flux through that surface is a measure of the total charge inside. This is the essence of Gauss's law.

- Now let's make it quantitative.

Module 2 (Electromagnetic Theory)

Electrostatics

- The Electric Flux

- The total flux due to a point charge Q: $\oint \boldsymbol{E}.d\boldsymbol{a} = \frac{Q}{4\pi\varepsilon_0}\int\frac{da}{\mathscr{r}^2}$ $\quad [\because \overline{\boldsymbol{E}} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}]$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{4\pi\mathscr{r}^2}{\mathscr{r}^2}$$

[$\because$ $\mathscr{r}$ is constant for the spherical surface and $\int da = 4\pi\mathscr{r}^2$]

$$= \frac{Q}{\varepsilon_0}$$

- Note: no. of field lines passing through the spherical surface = no. of field lines passing through ANY surface. Hence, $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q}{\varepsilon_0}$ (where *S* is an arbitrary surface enclosing Q).

Module 2 (Electromagnetic Theory)

Electrostatics

- The Electric Flux

- Now, suppose, there is a distribution of charge instead of a single charge at the origin.

- According to the principle of superposition, the total field is the vector sum of all the individual fields: $\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \boldsymbol{E}_i$

- The flux through a surface that encloses them all is: $\oint \boldsymbol{E}.\boldsymbol{da} = \sum_{i=1}^{n}\left(\oint \boldsymbol{E}_i.\boldsymbol{da}\right) = \sum_{i=1}^{n}\left(\frac{q_i}{\varepsilon_0}\right)$

- For any closed surface, then, $\oint_S \boldsymbol{E}.\boldsymbol{da} = \frac{Q_{enc}}{\varepsilon_0}$

- This is the **integral form of Gauss's law** or simply the Gauss's law.

Module 2 (Electromagnetic Theory)

Electrostatics

- The Electric Flux

- Integral form of Gauss's law for any closed surface: $\oint \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$

- We can easily turn it into a differential one, by applying the **divergence theorem**: $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \int_V (\boldsymbol{\nabla}.\boldsymbol{E})\, d\tau$

- Rewrite Q_{enc} in terms of the charge density ρ, we have: $Q_{enc} = \int_V \rho\, d\tau$

- Hence the integral form of Gauss's law ($\oint \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$) becomes: $\int_V (\boldsymbol{\nabla}.\boldsymbol{E})\, d\tau = \int_V (\frac{\rho}{\varepsilon_0})\, d\tau$

- Since it holds for any volume, the integrands must be equal: $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$

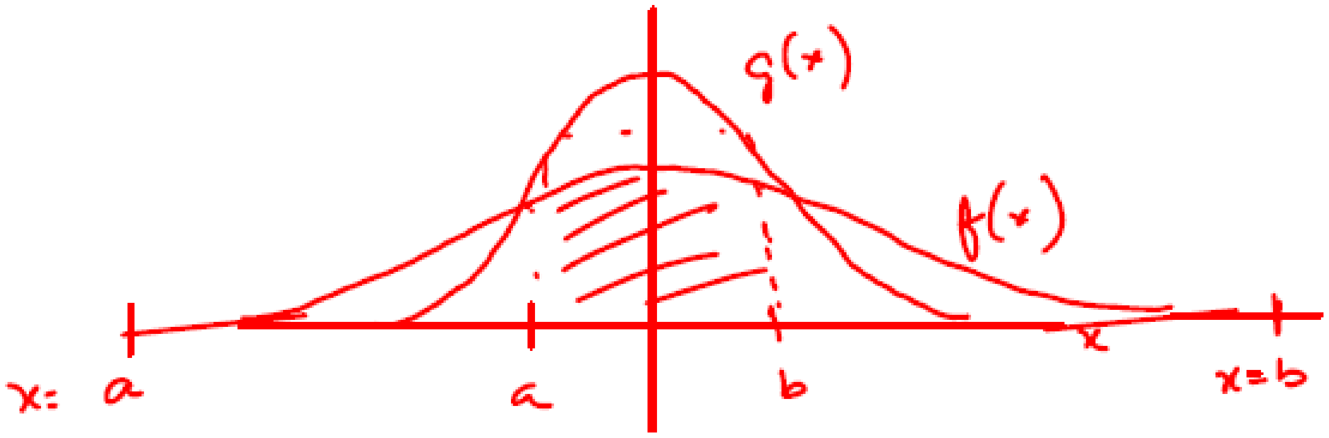
- This is the **differential form of Gauss's law**.

Module 2 (Electromagnetic Theory)

Electrostatics

- ❑ The Electric Flux

- ❑ Gauss's Law



- ❑ $\int_a^b g(x)dx = \int_a^b f(x)dx$

- ❑ These a and b are arbitrary

- ❑ $\int g(x)dx = \int f(x)dx$ is NOT possible for ANY given a, b unless the functions are same [i.e. unless $f(x) = g(x)$].

Module 2 (Electromagnetic Theory)

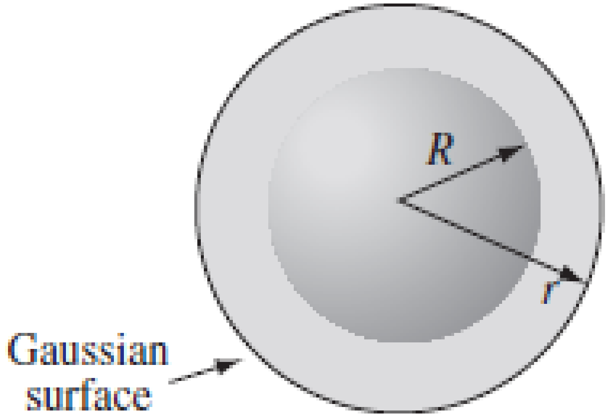
Electrostatics

- ❑ Application of Gauss's Law
- ❑ Gauss's Law
  - ❑ Integral form of Gauss's law: $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$
  - ❑ Differential form of Gauss's law: $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$
- ❑ When symmetry permits, Gauss's law in integral form affords by far the quickest and easiest way of computing electric fields.
- ❑ Solve some of the problems as example.
- ❑ Please go through the Assignment

Module 2 (Electromagnetic Theory)

Electrostatics

Application of Gauss's Law



Example 3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

- Consider a spherical surface at radius $r > R$ (This is called a Gaussian surface)
- Gauss's law: $\oint_S \mathbf{E}.d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$ (In this case $Q_{enc} = q$)
- Symmetry allows us to extract $\mathbf{E}$ from under the integral sign: Both $\mathbf{E}$ and $d\mathbf{a}$ points radially outward, so we can drop the dot product: $\int_S \mathbf{E}.d\mathbf{a} = \int_S |\mathbf{E}|d\mathbf{a}$
- The magnitude of $\mathbf{E}$ is constant over the gaussian surface – so it can come outside the integral.
- $\int_S \mathbf{E}.d\mathbf{a} = \int_S |\mathbf{E}|d\mathbf{a} = |\mathbf{E}| \int_S d\mathbf{a} = |\mathbf{E}|\, 4\pi r^2$

Module 2 (Electromagnetic Theory)

Electrostatics

Application of Gauss's Law

Example 3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

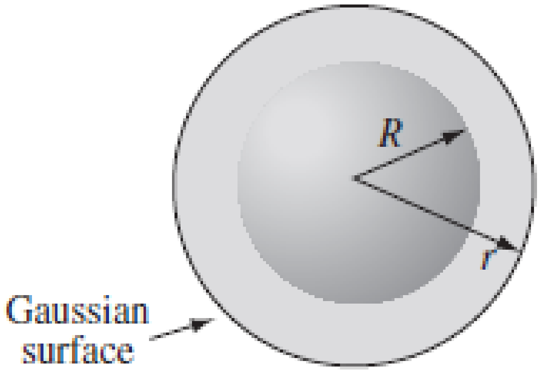


FIGURE 18

- Hence, $|\boldsymbol{E}|\, 4\pi r^2 = \frac{q}{\varepsilon_0}$ Or, $\overline{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}}$
- Note a remarkable feature of this result: The field outside the sphere is exactly the same as it would have been if all the charge had been concentrated at the center.

Module 2 (Electromagnetic Theory)

Electrostatics

Application of Gauss's Law

- Note: Gauss's law is always true, but it is not always useful.
  - In the previous example, $\boldsymbol{E}$ is pointed in the same direction as $d\boldsymbol{a}$ and its magnitude is constant over the surface. That's why we could take $|\boldsymbol{E}|$ outside the integral.
  - Unless we assume a spherically symmetrical shape (ρ must be uniform), this will not be valid. Hence, **Symmetry is crucial to this application of Gauss's law**.
- Three kinds of symmetry that work:
  1. Spherical symmetry (Gaussian surface is a concentric sphere).
  2. Cylindrical symmetry (Gaussian surface is a coaxial cylinder).
  3. Plane symmetry (Gaussian "pillbox" that straddles the surface).

Module 2 (Electromagnetic Theory)

Electrostatics

Application of Gauss's Law

Example 4. A long cylinder (Fig. 21) carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

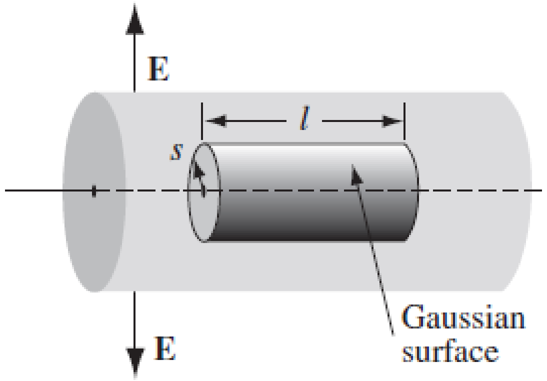
Solution

Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

The enclosed charge is

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi k l s^3.$$



(I used the volume element appropriate to cylindrical coordinates, and integrated ϕ from 0 to 2π, dz from 0 to l. I put a prime on the integration variable s', to distinguish it from the radius s of the Gaussian surface.)

Now, symmetry dictates that $\mathbf{E}$ must point radially outward, so for the curved portion of the Gaussian cylinder we have:

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l,$$

while the two ends contribute nothing (here $\mathbf{E}$ is perpendicular to $d\mathbf{a}$). Thus,

$$|\mathbf{E}| \, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3,$$

or, finally,

$$\mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$$

Electrostatics

Application of Gauss's Law

Example 5. An infinite plane carries a uniform surface charge σ. Find its electric field.

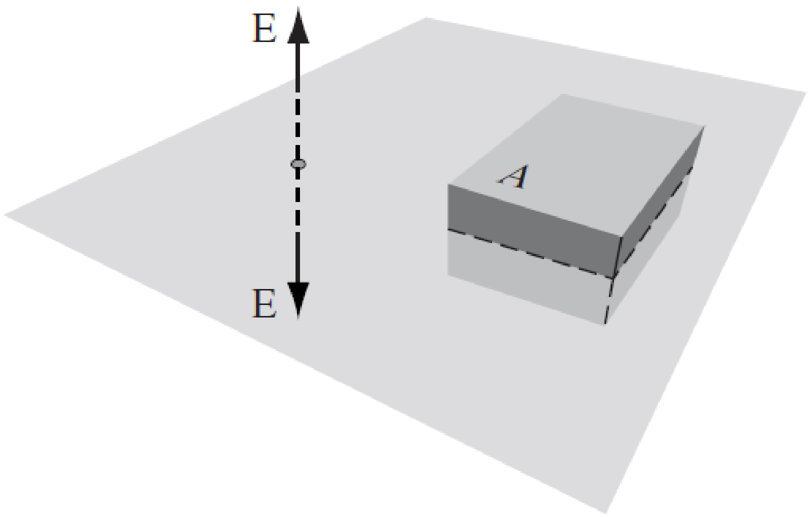


FIGURE 22

Solution

Draw a "Gaussian pillbox," extending equal distances above and below the plane (Fig. 22). Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\mathbf{E}$ points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$

whereas the sides contribute nothing. Thus

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0}\sigma A,$$

or

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}, \qquad (17)$$

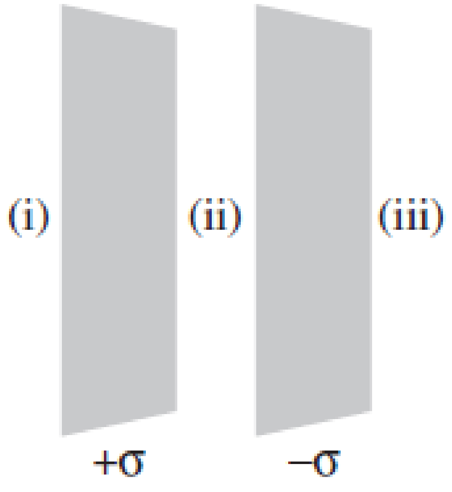
where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface. In Prob. 6, you obtained this same result by a much more laborious method.

Module 2 (Electromagnetic Theory)

Electrostatics

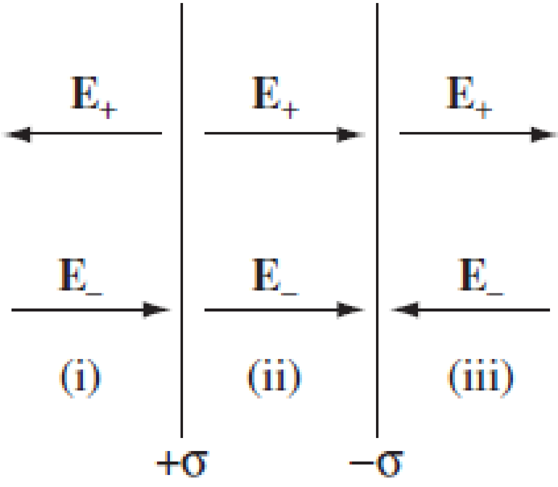
Application of Gauss's Law

Example 6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.



Solution

The left plate produces a field $(1/2\epsilon_0)\sigma$, which points away from it (Fig. 24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.



Module 2

Next Class

Questions?

PH113: Physics (B.Tech. I)

Module 2

Date: 29.09.2023

Lecture: 3

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

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Class structure: 4 Lectures including 1 Tutorial per week. (8 hours ~ 2 weeks for this module!)

Module 2 (Electromagnetic Theory)

Electrostatics

- **Curl of $\boldsymbol{E}$**
- For a point charge at origin: $\bar{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{r}}$
- Let's calculate the **line integral** of this field from some point *a* to some other point *b*: $\int_a^b \boldsymbol{E}.\boldsymbol{dl}$
- In spherical coordinates: $\boldsymbol{dl} = dr\,\hat{\boldsymbol{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\,sin\theta\,d\varphi\,\hat{\boldsymbol{\varphi}}$, so

$$\int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right),$$

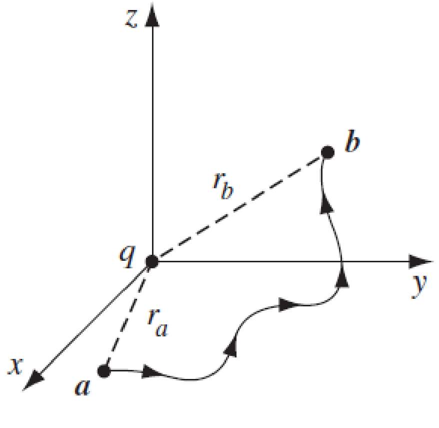


FIGURE 29

where r_a is the distance from the origin to the point *a* and r_b is the distance to *b*.

- The integral around a closed path is zero (for then $r_a = r_b$): $\oint_C \boldsymbol{E}.\boldsymbol{dl} = 0$
- Now, applying Stokes' theorem: $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$ ← irrotational field or conservative field

Module 2 (Electromagnetic Theory)

Electrostatics

Gauss's Law

- $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$
- $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$

Curl of $\boldsymbol{E}$

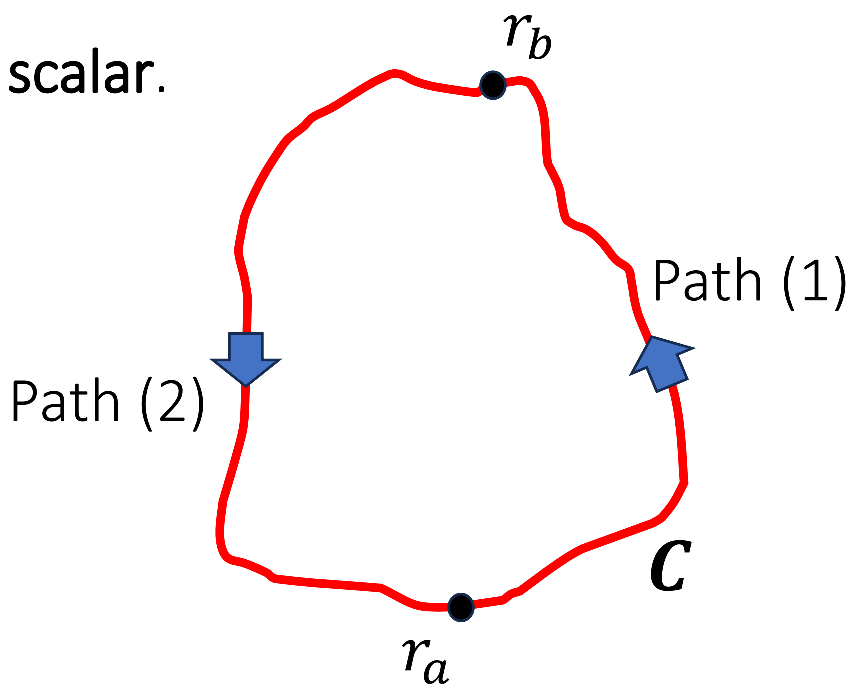
- $\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0$
- $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$

- $\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0$ implies that the line integral of $\boldsymbol{E}$ along a closed path must be zero. Physically, this means that no net work is done in moving a charge along a closed path in an electrostatic field.
- Applying Stokes theorem, we get: $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$
- Any vector field that satisfies these two equations is said to be conservative, or irrotational. In other words, vectors whose line integral does not depend on the path of integration are called conservative vector fields.
- Thus, an electrostatic field is a conservative field.

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Potential
- **Remember**: The electric field $\boldsymbol{E}$ is a very special kind of **vector** function whose curl is zero
- $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ ← irrotational field or conservative field
- Any vector whose **curl is zero** is equal to the **gradient of some scalar**.
- Because $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$, the line integral of $\boldsymbol{E}$ around any closed loop is zero (that follows from Stokes' theorem).
- Since curl of gradient is zero, $\boldsymbol{E} \propto \boldsymbol{\nabla} V$
- Because $\oint_C \boldsymbol{E}.\boldsymbol{dl} = 0$, the line integral of $\boldsymbol{E}$ from point a to point b is the same for all paths.



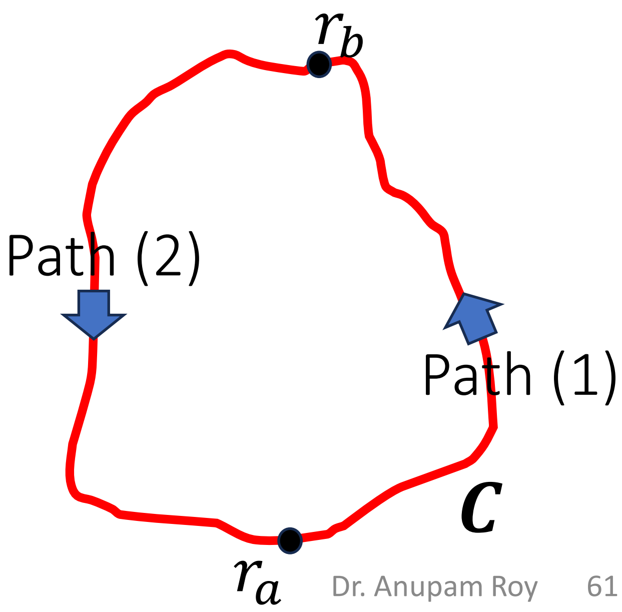
Module 2 (Electromagnetic Theory)

Electrostatics

Electric Potential

$$\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0 \quad => \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l} + \int_{\boldsymbol{r}_b}^{\boldsymbol{r}_a} \boldsymbol{E}.d\boldsymbol{l} = 0 \quad => \underbrace{\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}}_{\text{Path (1)}} = -\underbrace{\int_{\boldsymbol{r}_b}^{\boldsymbol{r}_a} \boldsymbol{E}.d\boldsymbol{l}}_{\text{Path (2)}} \quad => \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l} = \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}$$

- Hence, $\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}$ is **INDEPENDENT** of chosen path.



- Because the line integral is independent of path, we can define a function: $V(\boldsymbol{r}) \equiv -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}$ where $\boldsymbol{r}_0$ is the reference point. It is called the **electric potential**.

- Note: V depends only on the point $\boldsymbol{r}$.

Module 2 (Electromagnetic Theory)

Electrostatics

- **Electric Potential**

- The potential difference between two points *a* and *b* is

- $V(r_b) - V(r_a) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l} + \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_a} \boldsymbol{E}.d\boldsymbol{l} = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l} - \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_0} \boldsymbol{E}.d\boldsymbol{l} = -\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}$

- Now the fundamental theorem for gradients states that:

$V(r_b) - V(r_a) = \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} (\boldsymbol{\nabla} V).d\boldsymbol{l}$ [Remember: $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$]

- So, $\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} (\boldsymbol{\nabla} V).d\boldsymbol{l} = -\int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} \boldsymbol{E}.d\boldsymbol{l}$

- Since, this is true for any points *a* and *b*, the integrands must be equal: $\boldsymbol{E} = -\boldsymbol{\nabla} V$

- If V is known, one can easily get $\boldsymbol{E}$ just by taking the gradient: $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

Electrostatics

Electric Potential

- If V is known, one can easily get $\boldsymbol{E}$ just by taking the gradient: $\boldsymbol{E} = -\boldsymbol{\nabla} V$.
- $\boldsymbol{E}$ is a vector quantity (has three components), but V is a scalar (has only one component). How can one function possibly contain all the information that three independent functions carry?
- The answer is that the three components of $\boldsymbol{E}$ are not really as independent. They are related via $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$. In terms of components:

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}.$$

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Potential

- **Potential obeys the superposition principle**.

- The original superposition principle pertains to the force on a test charge Q. It says that the total force on Q is the vector sum of the forces attributable to the source charges individually: $\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2} + \ldots$

- Dividing through by Q, we see that the electric field, too, obeys the superposition principle (vector sum): $\boldsymbol{E} = \boldsymbol{E_1} + \boldsymbol{E_2} + \ldots$

- Integrating from the common reference point to r, it follows that the potential also satisfies such a principle (scalar sum): $V = V_1 + V_2 + \ldots$

- That is, the potential at any given point is the sum of the potentials due to all the source charges separately.

Module 2 (Electromagnetic Theory)

Electrostatics

- **Poisson's Equation and Laplace's Equation**

- Electric field can be written as the gradient of a scalar potential: $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

- Using $\boldsymbol{E} = -\boldsymbol{\nabla} V$ we get $\boldsymbol{\nabla}.\boldsymbol{E} = \boldsymbol{\nabla}.(-\boldsymbol{\nabla} V) = -\boldsymbol{\nabla}^2 V$

- (apart from the minus sign) Divergence of $\boldsymbol{E}$ is the Laplacian of $\boldsymbol{V}$.

- From the Gauss's law: $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$, we get: $\boldsymbol{\nabla}.(-\boldsymbol{\nabla} V) = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \boldsymbol{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$

- $\boldsymbol{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$ => This is known as Poisson's equation.

- In regions where there is no charge, we have $\rho = 0$, and Poisson's equation reduces to Laplace's equation: $\boldsymbol{\nabla}^2 V = 0$.

Module 2 (Electromagnetic Theory)

Electrostatics

- The Potential of a Localized Charge Distribution
- Section 3.4, Griffith (Please go through the textbook)
- Remember the expressions for $\boldsymbol{E}$?
  - The electric field of a line charge is: $\overline{\boldsymbol{E}}(\overline{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dl'}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$
  - For a surface charge: $\overline{\boldsymbol{E}}(\overline{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, da'}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$
  - For a volume charge: $\overline{\boldsymbol{E}}(\overline{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, d\tau'}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$
- Can you guess the expression for V?

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\, dl' \qquad V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\, da'. \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\, d\tau'.$$

Module 2 (Electromagnetic Theory)

Electrostatics

- The Work It Takes to Move a Charge

- Suppose you have a stationary configuration of source charges, and you want to move a test charge Q from point *a* to point *b*. How much work will you have to do?

- At any point along the path, the electric force on Q is $\boldsymbol{F} = Q\boldsymbol{E}$; the force you must exert, in opposition to this electrical force, is $-Q\boldsymbol{E}$.

- The work you do is therefore

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

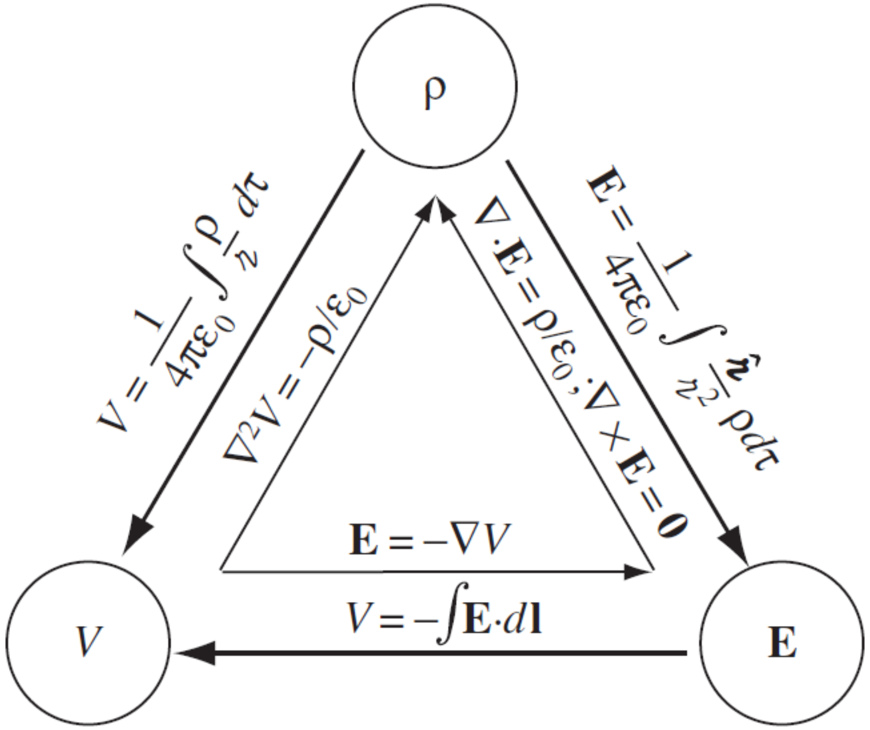
- Notice that the answer is independent of the path you take from *a* to *b*.

Module 2 (Electromagnetic Theory)

Electrostatics

- Three fundamental quantities of **electrostatics**: ρ, $\boldsymbol{E}$, and V.

- We have derived all six formulas interrelating them (summarized in the figure).



Electrostatics

❑ Electric Fields in Matter

- ❑ Matter can be broadly divided into two categories: conductors and insulators/dielectrics.
- ❑ Conductors contain an "unlimited" supply of charges that are free to move about through the material. Many of the electrons (one or two per atom, in a typical metal) are not associated with any particular nucleus, but roam around at will.
- ❑ In dielectrics, by contrast, all charges are attached (or, bound) to specific atoms or molecules.

Module 2 (Electromagnetic Theory)

Electrostatics

- ❑ Electric Fields in Matter

- ❑ **Induced Dipole**: What happens to a neutral atom when it is placed in an electric field $\boldsymbol{E}$?

- ❑ Positively charged core (nucleus) and negatively charged electron (e^-) surrounding it are influenced by the field: the nucleus is pushed in the direction of $\boldsymbol{E}$, and e^- the opposite way.

- ❑ Consider $\boldsymbol{E}$ is not too strong (so the atom is not ionized). The two opposing forces – $\boldsymbol{E}$ pulling the electrons and nucleus apart, their mutual attraction drawing them back together – reach a balance, leaving the atom **polarized**.

- ❑ The atom with separated +ve and –ve charges now has a tiny dipole moment $\boldsymbol{p}$, which points in the same direction as $\boldsymbol{E}$ and is approximately proportional to $\boldsymbol{E}$ (as long as $\boldsymbol{E}$ is not too strong).

- ❑ Dipole moment, $\boldsymbol{p} = \alpha\boldsymbol{E}$. (the constant of proportionality α is called atomic polarizability).

Module 2 (Electromagnetic Theory)

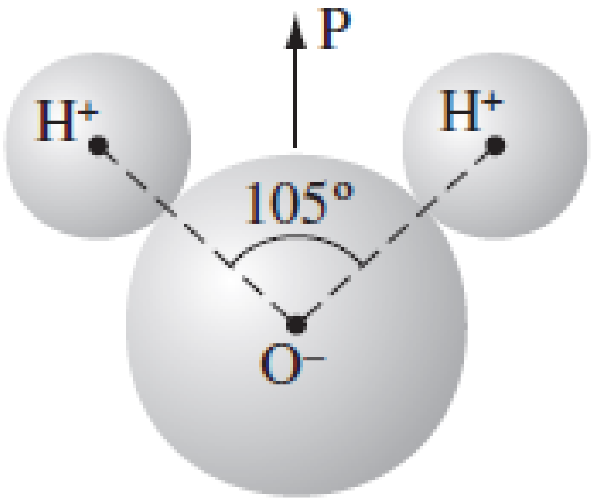
Electrostatics

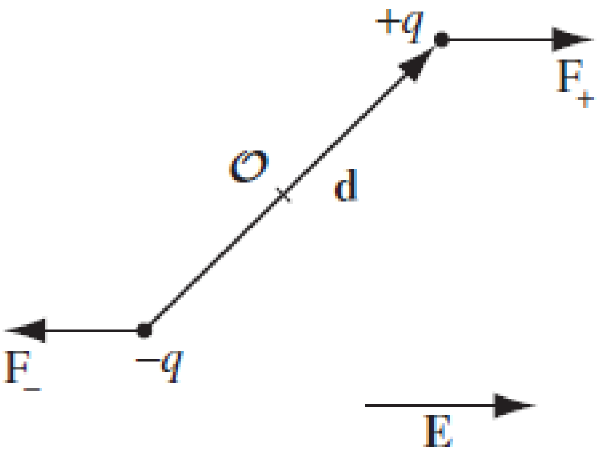
- Electric Fields in Matter

- **Induced Dipole**: What happens to a neutral atom when it is placed in an electric field $\boldsymbol{E}$?

- Neutral atom has no dipole moment to start with and $\boldsymbol{p}$ is induced by the applied $\boldsymbol{E}$ field.

- Some molecules have permanent dipole moments (e.g., water molecule) – **polar molecules**.

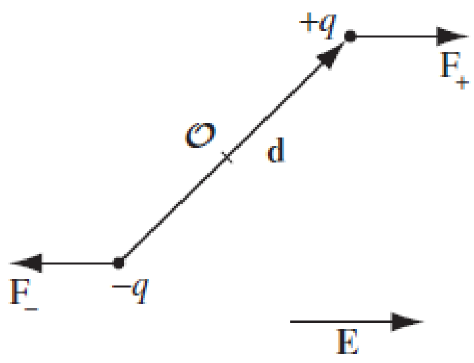




- For such molecules placed in a uniform electric field $\boldsymbol{E}$, the force on the positive end, $F_+ = q\boldsymbol{E}$, exactly cancels the force on the negative end, $F_- = -q\boldsymbol{E}$.

Module 2 (Electromagnetic Theory)

Electrostatics



- **Electric Fields in Matter**

- However, there will be a torque:

$$\boldsymbol{N} = (\boldsymbol{r}_+ \times \boldsymbol{F}_+) + (\boldsymbol{r}_- \times \boldsymbol{F}_-) = \left[\left(\frac{\boldsymbol{d}}{2}\right) \times (q\boldsymbol{E})\right] + \left[\left(-\frac{\boldsymbol{d}}{2}\right) \times (-q\boldsymbol{E})\right] = q\boldsymbol{d} \times \boldsymbol{E}$$

- Thus, a dipole $\boldsymbol{p} = q\boldsymbol{d}$ in a uniform field $\boldsymbol{E}$ experiences a torque $\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$.

- Notice that N is in such a direction as to line $\boldsymbol{p}$ up parallel to $\boldsymbol{E}$; in a polar molecule dipole moments get aligned along $\boldsymbol{E}$.

Dielectric material placed in an electric field, $\boldsymbol{E}$: *tiny dipoles point along the direction of* $\boldsymbol{E}$ (material becomes **polarized**). We define a vector called polarization vector, $\boldsymbol{P} \equiv$ *dipole moment per unit volume.*

Module 2 (Electromagnetic Theory)

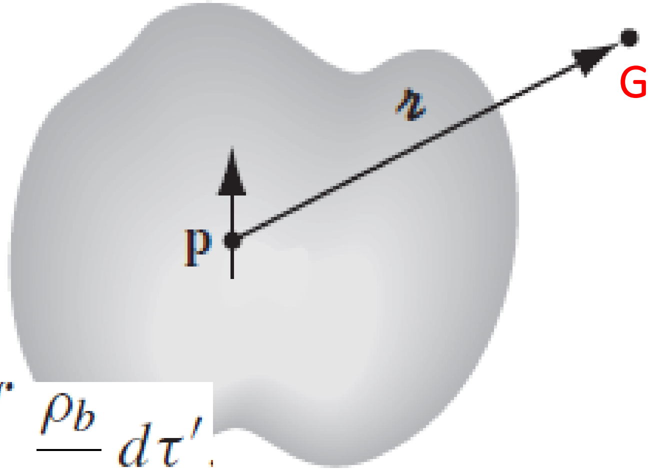
Electrostatics

- ❑ Electric Fields in Matter
- ❑ **Further Reading (Not in Syllabus)**
- ❑ What happens when $\boldsymbol{E}$ is applied to a conductor?

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Field of a Polarized Object



- Suppose we have a polarized material – an object containing a lot of microscopic dipoles lined up.
- What is the potential at a point G? $$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\,d\tau'.$$
- First integral is over the surface *S* enclosing the charge distribution ($\mathscr{r}$ is the position vector from a point on *S* to the point G). σ_b is the surface bound charge density and is given by $\boldsymbol{\sigma_b \equiv P\,.\,\hat{n}}$ (where $\hat{\boldsymbol{n}}$ is the normal unit vector).
- Second integral is over the volume *V* ($\mathscr{r}$ is the position vector from any volume element enclosed by *S* to the point G). ρ_b is the volume bound charge density and is given by $\boldsymbol{\rho_b \equiv -\nabla\,.\,P}$.
- For more details: Section 2, Chapter 4. Electric Fields in Matter, David J. Griffiths

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter
- Gauss's Law in the Presence of Dielectrics
- The effect of polarization is to produce accumulations of (bound) charge, $\rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P}$ within the dielectric and $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ on the surface.
- The field due to polarization of the medium is just the field of this **bound charge**.
- Also, the sample consists of **free charges** (electrons or ions).
- Hence, within the dielectric, the **total volume charge density** can be written as: $\rho = \rho_b + \rho_f$.

where, ρ_b is volume density of bound charges and ρ_f is volume density of free charges.

Module 2 (Electromagnetic Theory)

Electrostatics

Electric Fields in Matter: Gauss's Law in the Presence of Dielectrics

- In a dielectric, the total volume charge density can be written as: $\rho = \rho_b + \rho_f$.

- Gauss's law for $\boldsymbol{E}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_b + \rho_f}{\varepsilon_0}$$

$$\Rightarrow \quad \varepsilon_0 \, \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_b + \rho_f$$

$$\Rightarrow \quad \varepsilon_0 \, \boldsymbol{\nabla} \cdot \boldsymbol{E} = -\boldsymbol{\nabla} \cdot \boldsymbol{P} + \rho_f \qquad \text{(Since, } \rho_b = -\boldsymbol{\nabla} \cdot \boldsymbol{P}\text{)}$$

$$\Rightarrow \quad \boldsymbol{\nabla} \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_f$$

We define the Electric Displacement Vector as $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

Hence, $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$ (This is the differential form of Gauss's law in matter)

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Gauss's Law in the Presence of Dielectrics

- In a dielectric, the total volume charge density can be written as: $\rho = \rho_b + \rho_f$.

- Differential form of Gauss's law in matter: $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f$

Where the Electric Displacement Vector, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$

- Integral form of Gauss's law in matter:

$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{D})\, d\tau = \int_V \rho_f \, d\tau = Q_{f,enc}$ [$Q_{f,enc}$ is the enclosed free charge]

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter
- Curl of $\boldsymbol{D}$:
- The Electric Displacement Vector, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- Hence, $\boldsymbol{\nabla} \times \boldsymbol{D} = \varepsilon_0 (\boldsymbol{\nabla} \times \boldsymbol{E}) + \boldsymbol{\nabla} \times \boldsymbol{P}$
- But $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$
- So, $\boldsymbol{\nabla} \times \boldsymbol{D} = \boldsymbol{\nabla} \times \boldsymbol{P}$
- Note: For the conservative field: (i) $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ and (ii) $\boldsymbol{E} = -\boldsymbol{\nabla} V$.

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Linear Dielectric
- If $\boldsymbol{E}$ is not too strong, polarization is proportional to the field:

$$\boldsymbol{P} = \alpha \boldsymbol{E}$$

$$= \varepsilon_0 \, \chi_e \, \boldsymbol{E}$$ (where $\chi_e = \alpha/\varepsilon_0$ is the **electric susceptibility** of the medium)

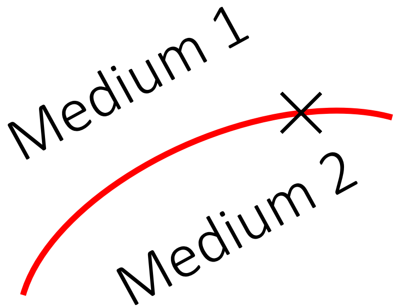
- Materials that obey the relation $\boldsymbol{P} = \varepsilon_0 \, \chi_e \, \boldsymbol{E}$ are called linear dielectric materials.
- The Electric Displacement Vector, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \, \chi_e \, \boldsymbol{E} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon \, \boldsymbol{E}$
- ε is the permittivity of the material and ε_0 is the permittivity of vacuum (or free space).
- So, $\boldsymbol{D}$ is also proportional to $\boldsymbol{E}$ and it can be written as $\boldsymbol{D} = \varepsilon \, \boldsymbol{E}$ [where $\varepsilon = \varepsilon_0 (1 + \chi_e)$]
- Relative permittivity (or dielectric constant) is defined as: $\varepsilon_r = 1 + \chi_e = \varepsilon/\varepsilon_0$

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

- What happens to the parallel and perpendicular components of $\boldsymbol{E}$ and $\boldsymbol{D}$ when they cross an interface?



- Decompose the electric field $\boldsymbol{E}$ vector into two orthogonal components: $\boldsymbol{E} = \boldsymbol{E}_{\perp} + \boldsymbol{E}_{\parallel}$ where $\boldsymbol{E}_{\perp}$ and $\boldsymbol{E}_{\parallel}$ are the **normal** and **tangential** components of $\boldsymbol{E}$ to the interface, respectively.
- A similar decomposition can be done for $\boldsymbol{D}$.

1. Boundary condition for $\boldsymbol{E}$:
   a. Normal (perpendicular) component
   b. Tangential (parallel) component

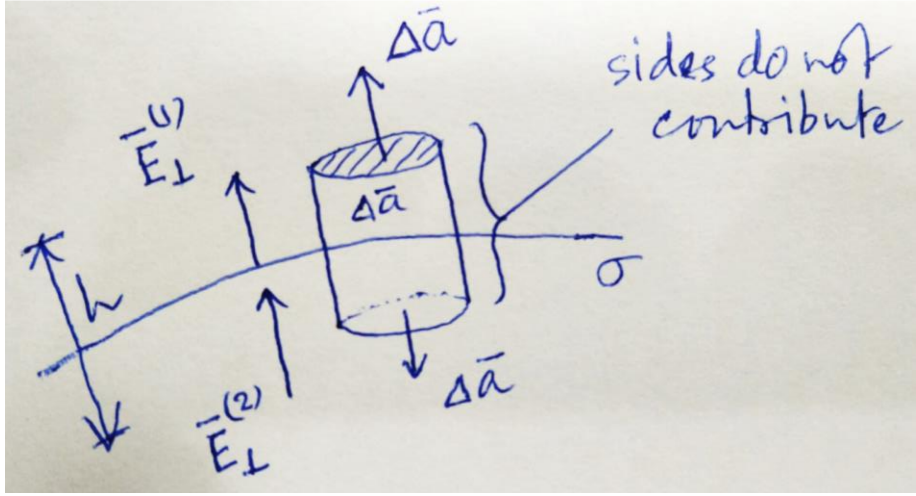
2. Boundary condition for $\boldsymbol{D}$:
   a. Normal (perpendicular) component
   b. Tangential (parallel) component

Module 2 (Electromagnetic Theory)

Electrostatics

❑ Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

1. Boundary condition for $\boldsymbol{E}$: (a) Normal Component:



❑ We consider a cylindrical Gaussian pillbox of surface S with a **small** circular cross-section Δa and height $\boldsymbol{h}$

❑ From the Gauss's law: $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0} = \oint_S \frac{\sigma\, da}{\varepsilon_0}$

Where σ is the surface charge density and ε_0 is the permittivity of free space.

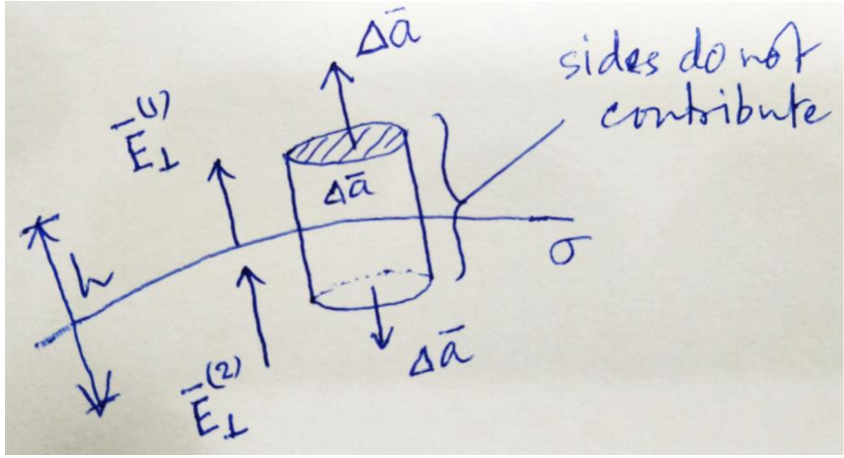
❑ So, $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \oint_{upper} \boldsymbol{E}.d\boldsymbol{a} + \oint_{lower} \boldsymbol{E}.d\boldsymbol{a} + \oint_{sides} \boldsymbol{E}.d\boldsymbol{a}$

In the limit $h \to 0$, sides of the pillbox do not contribute, and we have $\oint_{sides} \boldsymbol{E}.d\boldsymbol{a} = 0$.

Module 2 (Electromagnetic Theory)

Electrostatics

❑ Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$



1. Boundary condition for $\boldsymbol{E}$: (a) Normal Component:

❑ $\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0} = \oint_S \frac{\sigma\, da}{\varepsilon_0}$

$$\oint_S \boldsymbol{E}.d\boldsymbol{a} = \oint_{upper} \boldsymbol{E}.d\boldsymbol{a} + \oint_{lower} \boldsymbol{E}.d\boldsymbol{a} = \oint_S \frac{\sigma\, da}{\varepsilon_0}$$

$$E_{\perp}^{(1)}\,\Delta a + E_{\perp}^{(2)}\,(-\Delta a) = \frac{\sigma\,\Delta a}{\varepsilon_0}$$

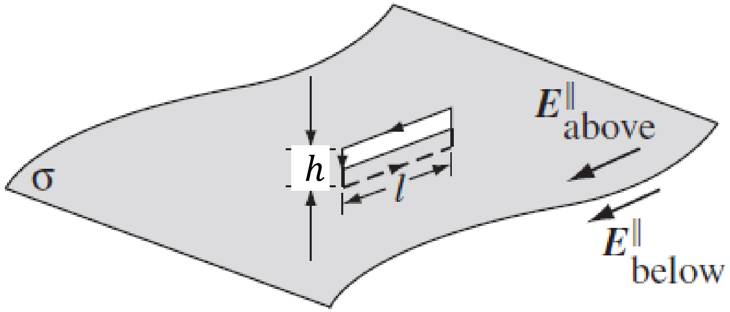
$$\Rightarrow\ E_{\perp}^{(1)} - E_{\perp}^{(2)} = \frac{\sigma}{\varepsilon_0} \neq 0$$

Conclusion: In presence of a **finite** surface charge density $\sigma \neq 0$, the **normal component** of $\boldsymbol{E}$ is discontinuous across the interface by an amount of $\frac{\sigma}{\varepsilon_0}$. (Where there is no surface charge, $\boldsymbol{E}_{\perp}$ is continuous, as for instance at the surface of a uniformly charged solid sphere.)

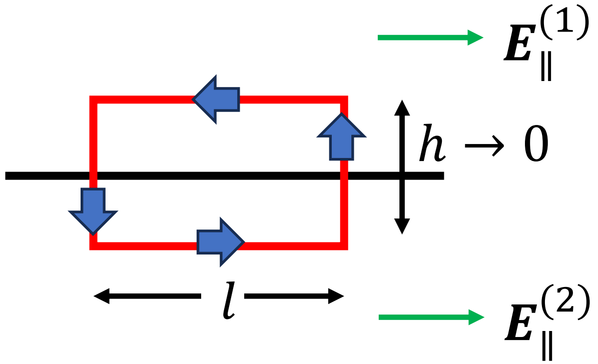
Electrostatics

❑ Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

1. Boundary condition for $\boldsymbol{E}$: (b) Tangential Component:



- ❑ $\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0$
- ❑ Again, consider the height, $h \to 0$. Hence, only the longer sides contribute (shorter sides of length $h \to 0$ do not contribute)



- ❑ $E_{\parallel}^{(1)}\,(-l) + E_{\parallel}^{(2)}\,(l) = 0$
- ❑ $E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$

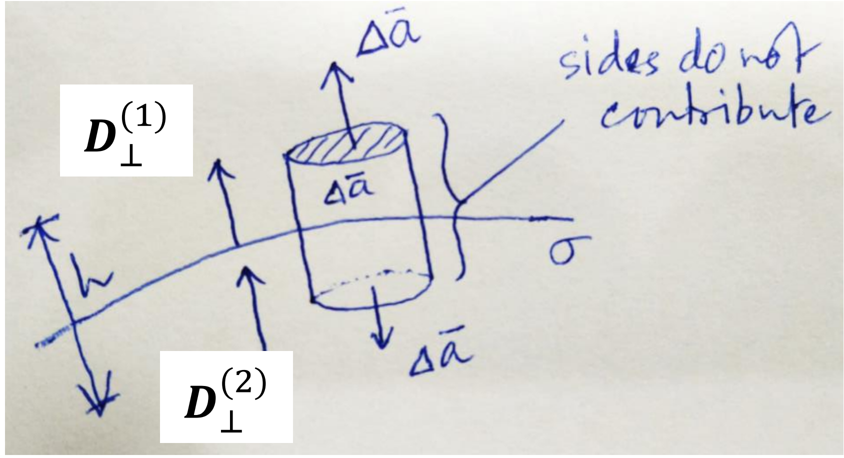
❑ **Conclusion**: **Tangential component** of $\boldsymbol{E}$, by contrast, is always continuous across an interface.

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

2. Boundary condition for $\boldsymbol{D}$: (a) Normal Component:



- $\oint_S \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f,\,enc} = \oint_S \sigma_f \, da$

$$D_\perp^{(1)} \, \Delta a + D_\perp^{(2)} \, (-\Delta a) = \sigma_f \, \Delta a$$

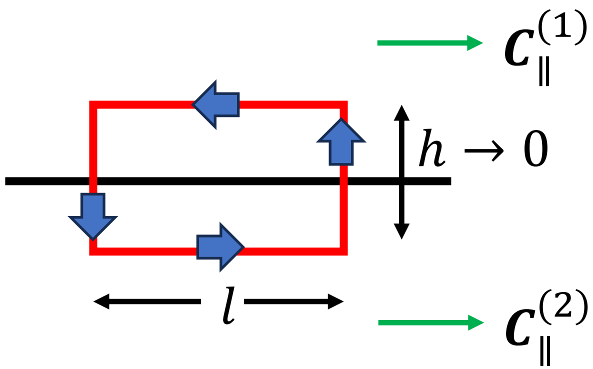
$$\Rightarrow \; D_\perp^{(1)} - D_\perp^{(2)} = \sigma_f \neq 0$$

Conclusion: In presence of a **finite** free surface charge density $\sigma_f \neq 0$, the **normal component** of $\boldsymbol{D}$ is discontinuous across the interface by an amount of σ_f.

Module 2 (Electromagnetic Theory)

Electrostatics

- Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$



2. Boundary condition for $\boldsymbol{D}$: (b) Tangential Component:

- $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$
- $\oint_C \boldsymbol{D}.d\boldsymbol{l} = \varepsilon_0 \oint_C \boldsymbol{E}.d\boldsymbol{l} + \oint_C \boldsymbol{P}.d\boldsymbol{l}$
- Since, $\oint_C \boldsymbol{E}.d\boldsymbol{l} = 0$, we have $\oint_C \boldsymbol{D}.d\boldsymbol{l} = \oint_C \boldsymbol{P}.d\boldsymbol{l}$
- Hence, $\oint_C (\boldsymbol{D} - \boldsymbol{P}).d\boldsymbol{l} = 0$
- Rewrite this as: $\oint_C \boldsymbol{C}.d\boldsymbol{l} = 0$ (where $\boldsymbol{C} = \boldsymbol{D} - \boldsymbol{P}$)

Module 2 (Electromagnetic Theory)

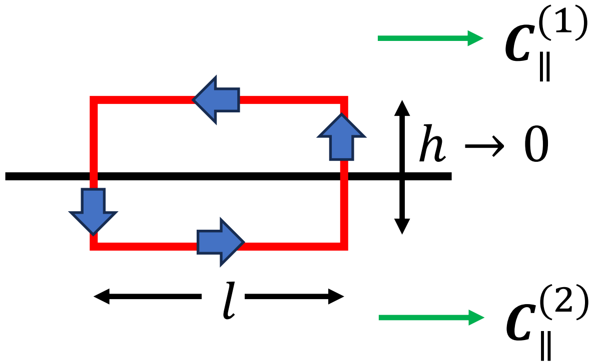
Electrostatics

❑ Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

2. Boundary condition for $\boldsymbol{D}$: (b) Tangential Component:

❑ Again, consider the height, $h \to 0$. Hence, only the longer sides contribute (shorter sides of length $h \to 0$ do not contribute). Similar to previous case for $\boldsymbol{E}$, we can have

❑ $C_\parallel^{(1)}\,(-l) + C_\parallel^{(2)}\,(l) = 0 \quad \Rightarrow \quad C_\parallel^{(1)} = C_\parallel^{(2)}$



❑ Hence, $D_\parallel^{(1)} - P_\parallel^{(1)} = D_\parallel^{(2)} - P_\parallel^{(2)}$

❑ This gives, $D_\parallel^{(1)} - D_\parallel^{(2)} = P_\parallel^{(1)} - P_\parallel^{(2)}$

❑ **Conclusion**: **Tangential component** of $\boldsymbol{D}$ is in general discontinuous across an interface unless $P_\parallel^{(1)} = P_\parallel^{(2)}$.

Electrostatics

❑ Electric Fields in Matter: Boundary Conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$

1. Boundary condition for $\boldsymbol{E}$:

   (a) Normal Component: $E_{\perp}^{(1)} - E_{\perp}^{(2)} = \frac{\sigma}{\varepsilon_0}$

   (b) Tangential Component: $E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$

2. Boundary condition for $\boldsymbol{D}$:

   (a) Normal Component: $D_{\perp}^{(1)} - D_{\perp}^{(2)} = \sigma_f$

   (b) Tangential Component: $D_{\parallel}^{(1)} - D_{\parallel}^{(2)} = P_{\parallel}^{(1)} - P_{\parallel}^{(2)}$

Next Class

Questions?

Module 2

Date: 03.10.2023

Lecture: 4

PH113: Physics (B.Tech. I)

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

Text Book: Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press (Seventh Edition, 2018)

Reference Book: David J. Griffiths, Introduction to Electrodynamics, Pearson (Fourth Edition, 2014)

Class structure: 4 Lectures including 1 Tutorial per week. (8 hours ~ 2 weeks for this module!)

Magnetostatics

❑ Lorentz Force Law

- ❑ A stationary charge produces only an electric field $\boldsymbol{E}$ in the space around it.
- ❑ However, a moving charge generates, in addition, a magnetic field $\boldsymbol{B}$.

Stationary charges	$\Rightarrow$	**constant electric fields: electrostatics.**
Steady currents	$\Rightarrow$	**constant magnetic fields: magnetostatics.**

- ❑ In the presence of both electric and magnetic fields, the net force on a charge Q moving with a velocity $\boldsymbol{v}$ would be $\boldsymbol{F} = Q[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]$.
- ❑ This is known as the Lorentz force law.
- ❑ In absence of electric field $\boldsymbol{E}$, we only have the magnetic force, $\boldsymbol{F}_{mag} = Q(\boldsymbol{v} \times \boldsymbol{B})$.

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Lorentz Force Law

- ❑ In the presence of both electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields, the net force on a charge Q moving with a velocity $\boldsymbol{v}$ would be $\boldsymbol{F} = Q[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})]$. This is known as the Lorentz force law.
- ❑ In absence of electric field $\boldsymbol{E}$, we only have the magnetic force, $\boldsymbol{F}_{mag} = Q(\boldsymbol{v} \times \boldsymbol{B})$.
- ❑ Note: Magnetic forces do no work.
- ❑ For if Q moves an amount $d\boldsymbol{l} = \boldsymbol{v}\, dt$, the work done is $dW_{mag} = \boldsymbol{F}_{mag} \cdot d\boldsymbol{l}$

$$= Q(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}\, dt = 0.$$

- Because $(\boldsymbol{v} \times \boldsymbol{B})$ is perpendicular to $\boldsymbol{v}$, so $(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} = 0$.

- ❑ Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.

Module 2 (Electromagnetic Theory)

Magnetostatics

- Lorentz Force Law

- **Current**: The current in a wire is the charge per unit time passing a given point.

- The magnetic force on a segment of current-carrying wire is

$$\boldsymbol{F}_{mag} = \int dq\,(\boldsymbol{v} \times \boldsymbol{B}) = \int I\,dt\,(\boldsymbol{v} \times \boldsymbol{B}) = \int I\,(d\boldsymbol{l} \times \boldsymbol{B}) \quad \text{(since } d\boldsymbol{l} = \boldsymbol{v}\,dt\text{)}$$

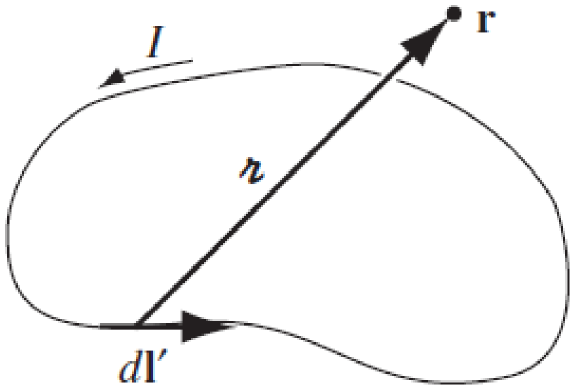
- Hence, $\boldsymbol{F}_{mag} = I \int (d\boldsymbol{l} \times \boldsymbol{B})$. (Typically, the current is a constant in magnitude along the wire)

- Suppose both $\boldsymbol{I}$ and $\boldsymbol{B}$ are constants. Then $\boldsymbol{F}_{mag} = I\,(\int (d\boldsymbol{l}) \times \boldsymbol{B} = I\,(\boldsymbol{l} \times \boldsymbol{B})$.

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Biot Savart Law



- ❑ The magnetic field of a **steady line current** is given by the Biot-Savart law: $\boldsymbol{B(r)} = \frac{\mu_0}{4\pi} \int I \frac{d\boldsymbol{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

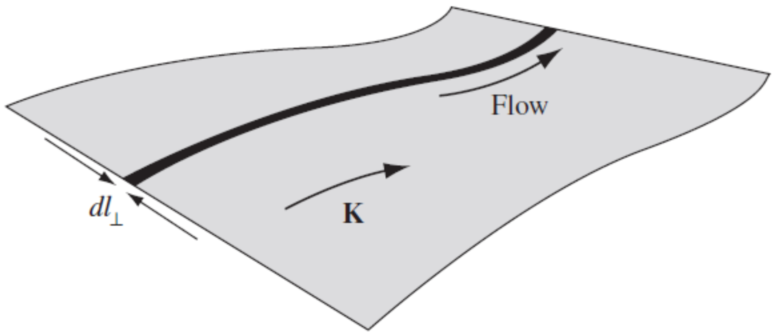
- ❑ The integration is along the current path, in the direction of the flow; $\boldsymbol{dl'}$ is an element of length along the wire, and $\boldsymbol{\mathscr{r}}$, as always, is the vector from the source to the point **r** (see figure).

- ❑ The constant μ_0 is called the **permeability** of free space and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2.

Magnetostatics

❑ Biot Savart Law

❑ When charge flows over a surface, we have a **surface current density** ($\boldsymbol{K}$), defined as the current per unit width perpendicular to the flow.



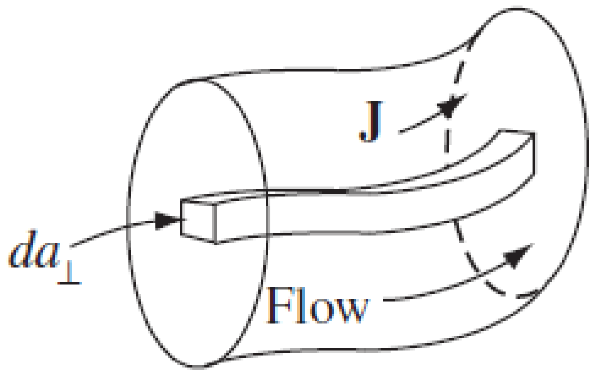
❑ Consider a "ribbon" of infinitesimal width $\boldsymbol{dl_\perp}$, running parallel to the flow (see Figure). If the current in this ribbon is $\boldsymbol{dI}$, the surface current density is $\boldsymbol{K} = \frac{\boldsymbol{dI}}{\boldsymbol{dl_\perp}}$.

❑ If the (mobile) surface charge density is $\boldsymbol{\sigma}$ and its velocity is $\boldsymbol{v}$, then $\boldsymbol{K} = \boldsymbol{\sigma v}$.

❑ For a **surface current**, Biot-Savart law becomes: $\boldsymbol{B(r)} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{K(r')} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da'$

Magnetostatics

❑ Biot Savart Law



- ❑ When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density** ($\boldsymbol{J}$), defined as the current per unit area.

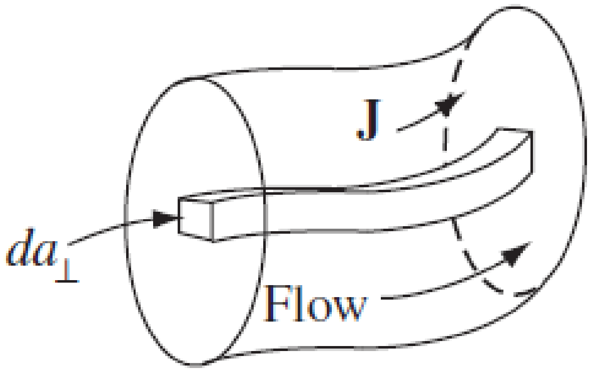
- ❑ Consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow (see Figure). If the current in this tube is $d\boldsymbol{I}$, the volume current density is: $\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_\perp}$.

- ❑ If the (mobile) volume charge density is ρ and the velocity is v, then $\boldsymbol{J} = \rho\boldsymbol{v}$.

- ❑ For a volume current, Biot-Savart law becomes: $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'$

Magnetostatics

❑ Continuity Equation



- Consider a "tube" of infinitesimal cross section $da_\perp$, running parallel to the flow (see Figure). If the current in this tube is $d\boldsymbol{I}$, the volume current density is: $\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_\perp}$.

- Total current crossing a surface *S* can be written as: $I = \int_S J\, da_\perp = \int_S \boldsymbol{J} \cdot d\boldsymbol{a}$

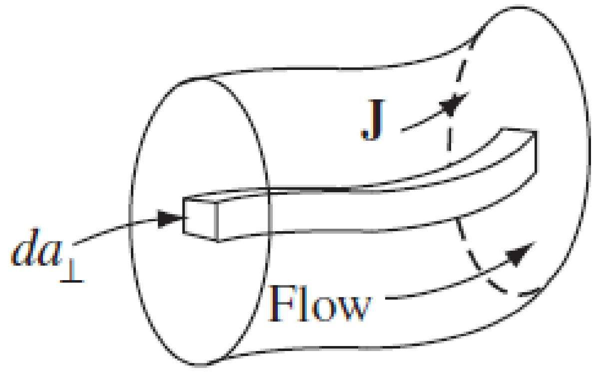
(The dot product serves neatly to pick out the appropriate component of $d\boldsymbol{a}$.)

- Charge per unit time leaving a volume *V* is: $\int_S \boldsymbol{J} \cdot d\boldsymbol{a} = \int_V (\boldsymbol{\nabla} \cdot \boldsymbol{J})\, d\tau$

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Continuity Equation



❑ Consider a “tube” of infinitesimal cross section $\boldsymbol{da}$, running parallel to the flow. If the volume current density is $\boldsymbol{J}$, the total current crossing a surface *S* can be written as: $I = \int_S J \, da_\perp = \int_S \boldsymbol{J} . d\boldsymbol{a} = \int_V (\boldsymbol{\nabla} . \boldsymbol{J}) \, d\tau$

❑ Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside: $\int_V (\boldsymbol{\nabla} . \boldsymbol{J}) \, d\tau = -\frac{d}{dt}\int_V \rho \, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$

(The minus sign reflects the fact that an outward flow decreases the charge left in *V*.)

❑ Since this applies to any volume, we conclude that: $\boldsymbol{\nabla} . \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$

❑ This is the precise mathematical statement of local charge conservation and is called the **continuity equation**.

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Biot Savart Law

Homework

Section 2.2 (Magnetostatics) from Griffith

Example 5. Find the magnetic field a distance s from a long straight wire carrying a steady current I (Fig. 18).

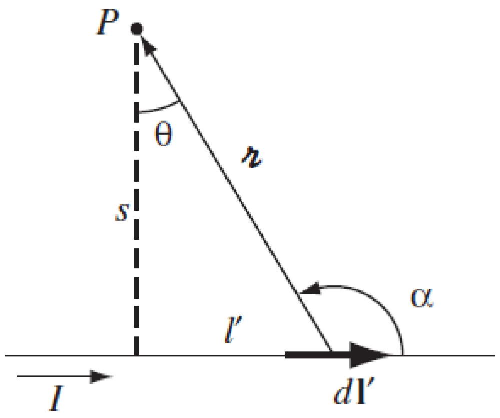


FIGURE 18

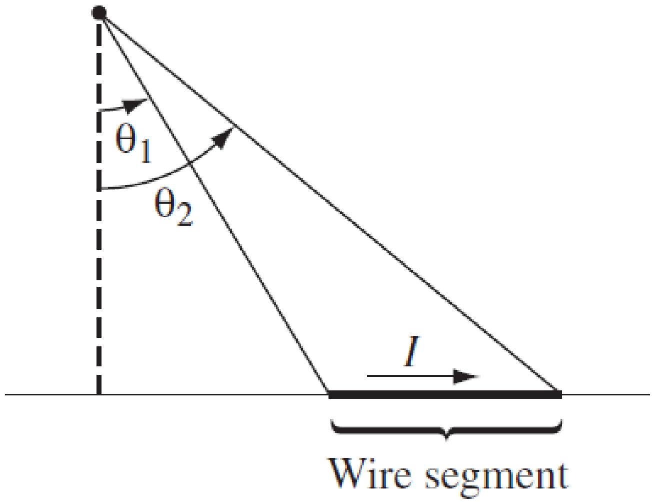


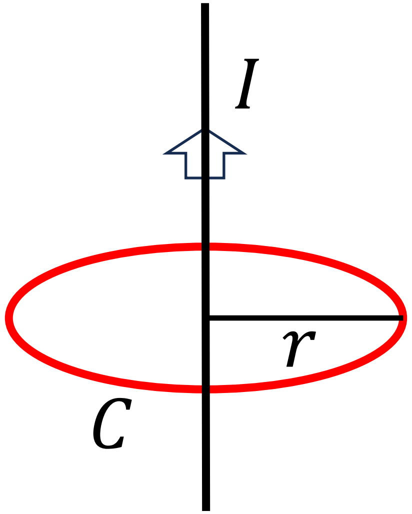
FIGURE 19

- What is the magnetic field by an **infinite** straight current carrying wire at a distance s?
  - $B = \frac{\mu_0 I}{2\pi s}$ (the field is inversely proportional to the distance from the wire)

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Curl of $\boldsymbol{B}$



- ❑ The integral of $\boldsymbol{B}$ around a circular path C of radius r, centered at the wire
- ❑ $\int_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \,.\, d\boldsymbol{a} = \oint_C \boldsymbol{B} \,.\, d\boldsymbol{l}$ (using Stokes' theorem)

$$= \frac{\mu_0 I}{2\pi r} \oint dl = \frac{\mu_0 I}{2\pi r} (2\pi r) = \mu_0 I$$

- ❑ Suppose we have a bundle of straight wires. Each wire that passes through the loop contributes $\mu_0 I$, and those outside contribute nothing. The line integral will then be

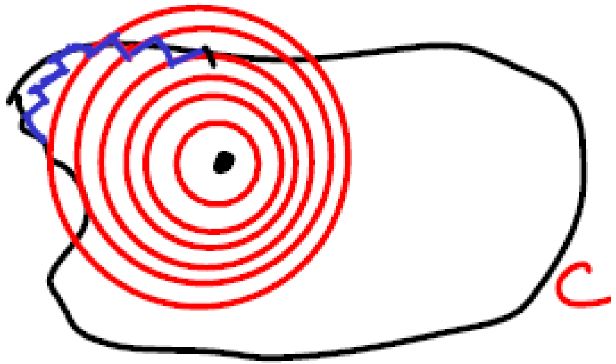
$$\oint_C \boldsymbol{B} \,.\, d\boldsymbol{l} = \mu_0 I_{enc}$$ (where I_{enc} is the total current enclosed by the loop C)

- ❑ **Note**: the result is independent of r. Hence it does not have to be a circle; it is applicable to any shape of closed loop.

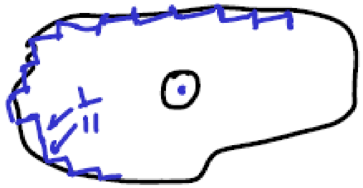
Module 2 (Electromagnetic Theory)

Magnetostatics

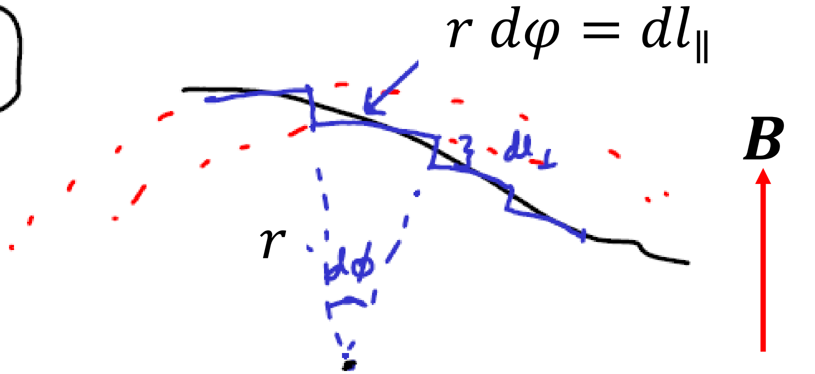
❑ Curl of $\boldsymbol{B}$



- ❑ The entire loop C can be represented by an infinitesimal line element **along** one of the concentric circles and an infinitesimal element perpendicular to that circle.



- ❑ $\oint_C \boldsymbol{B}\,.\,d\boldsymbol{l} = \oint_C B\,r\,d\varphi$ ($\because$ line element that are perpendicular to concentric circles do not contribute to the integral).

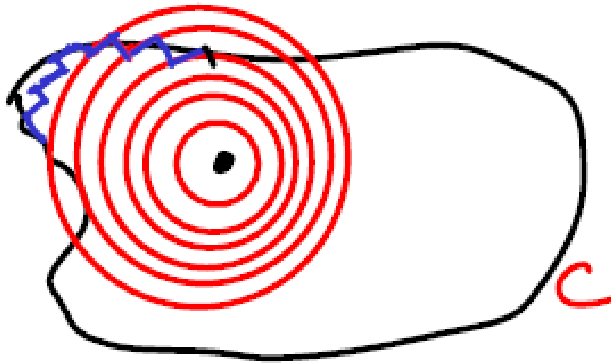


- $d\boldsymbol{l} = d\boldsymbol{l}_{\parallel} + d\boldsymbol{l}_{\perp}$
- $\boldsymbol{B}\,.\,d\boldsymbol{l} = B dl_{\parallel}$

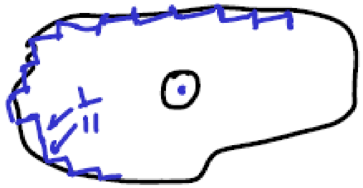
Module 2 (Electromagnetic Theory)

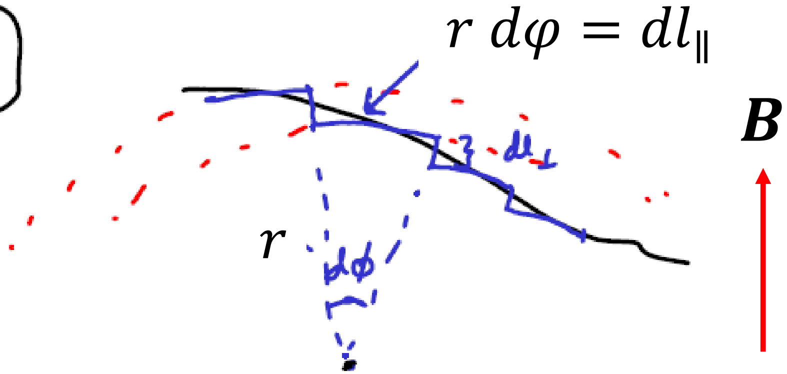
Magnetostatics

❑ Curl of $\boldsymbol{B}$



- ❑ For any concentric circle of radius r, magnetic field produced is $\frac{\mu_0\, I}{2\pi r}$ **along** the direction of $\boldsymbol{dl}_{\parallel}$ which lies on the circumference of that circle.





- ❑ Hence, $\oint_C \boldsymbol{B}\,.\,\boldsymbol{dl} = \oint_C B\, r\, d\varphi = \oint_C \frac{\mu_0\, I}{2\pi r}\, r\, d\varphi$

$$= \frac{\mu_0\, I_{enc}}{2\pi}\oint_C d\varphi = \frac{\mu_0\, I_{enc}}{2\pi}\,(2\pi)$$

- ❑ Hence, $\oint_C \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0\, I_{enc}$
- ❑ This is the integral form of Ampere's Law.

- $\boldsymbol{dl} = \boldsymbol{dl}_{\parallel} + \boldsymbol{dl}_{\perp}$
- $\boldsymbol{B}\,.\,\boldsymbol{dl} = B\, dl_{\parallel}$

Magnetostatics

Curl of $\boldsymbol{B}$

- If the flow of charge is represented by a volume current density $\boldsymbol{J}$, the enclosed current is $I_{enc} = \int_S \boldsymbol{J} \,.\, d\boldsymbol{a}$. [since $\boldsymbol{J} = \frac{d\boldsymbol{I}}{da_{\perp}} = \frac{d\boldsymbol{I}}{da\, cos\theta}$ and the integral is taken over any surface bounded by the loop]

- Apply Stokes' theorem: $\int_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \,.\, d\boldsymbol{a} = \oint_C \boldsymbol{B} \,.\, d\boldsymbol{l} = \mu_0\, I_{enc} = \mu_0 \int_S \boldsymbol{J} \,.\, d\boldsymbol{a}$

- Hence, $\int_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \,.\, d\boldsymbol{a} = \mu_0 \int_S \boldsymbol{J} \,.\, d\boldsymbol{a}$

- Since S is arbitrary, we have: $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0\, \boldsymbol{J}$

- This is the relation for the curl of $\boldsymbol{B}$. This is also known as the differential form of Ampere's Law.

Magnetostatics

Divergence of $\boldsymbol{B}$

- Since the number of field lines exiting the surface equals the number of field lines entering the surface, we have: $\oint_S \boldsymbol{B} \,.\, d\boldsymbol{a} = 0$
- This is because the magnetic fields always form closed loops unlike electric fields.
- Since, $\oint_S \boldsymbol{B} \,.\, d\boldsymbol{a} = 0$, we have $\oint_S \boldsymbol{B} \,.\, d\boldsymbol{a} = \int_V (\boldsymbol{\nabla} \,.\, \boldsymbol{B})\, d\tau = 0$
- Hence, $\boldsymbol{\nabla} \,.\, \boldsymbol{B} = 0$

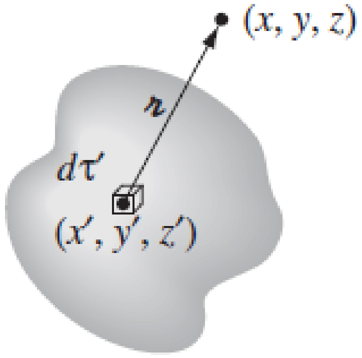
Curl of $\boldsymbol{B}$

- $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$, Or
- $\oint_C \boldsymbol{B} \,.\, d\boldsymbol{l} = \mu_0 I_{enc}$

Divergence of $\boldsymbol{B}$

- $\boldsymbol{\nabla} \,.\, \boldsymbol{B} = 0$, Or
- $\oint_S \boldsymbol{B} \,.\, d\boldsymbol{a} = 0$

Module 2 (Electromagnetic Theory)



Magnetostatics

- Derivation of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

- The Biot-Savart law for a volume current: $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$

- Note:
  - This formula gives the magnetic field ($\boldsymbol{B}$) at a point $\boldsymbol{r} = (x, y, z)$ in terms of an integral over the current distribution $\boldsymbol{J}(x', y', z')$.
  - So, $\boldsymbol{B}$ is a function of (x, y, z), $\boldsymbol{J}$ is a function of (x', y', z'), $\boldsymbol{\mathscr{r}} = (x - x')\hat{\boldsymbol{x}} + (y - y')\hat{\boldsymbol{y}} + (z - z')\hat{\boldsymbol{z}}$ and $d\tau' = dx'dy'dz'$.
  - The integration is over the primed coordinates; the divergence and the curl of $\boldsymbol{B}$ are with respect to the unprimed coordinates.

Module 2 (Electromagnetic Theory)

Magnetostatics

- **Derivation of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$**

- The Biot-Savart law for a volume current: $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$

- Take divergence of both sides: $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \cdot \left(\boldsymbol{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau'$

- Now use the vector calculus identity: $\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$ (prove it)

- Then we have $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\boldsymbol{\nabla} \times \boldsymbol{J}) - \boldsymbol{J} \cdot \left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right] d\tau'$

- Now, $\boldsymbol{J} \equiv \boldsymbol{J}(x', y', z')$ is a function of primed coordinates. But $\boldsymbol{\nabla} \equiv \hat{\boldsymbol{x}}\left(\frac{\partial}{\partial x}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial}{\partial y}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial}{\partial z}\right)$ (unprimed coordinates)

- Since $\boldsymbol{J}$ is independent of the unprimed coordinates, $\boldsymbol{\nabla} \times \boldsymbol{J} = 0$

Module 2 (Electromagnetic Theory)

Magnetostatics

- Derivation of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

- The Biot-Savart law for a volume current: $\boldsymbol{B(r)} = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J(r')} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$

- Take divergence: $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{\mu_0}{4\pi} \int \boldsymbol{\nabla} \cdot \left(\boldsymbol{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau' = \frac{\mu_0}{4\pi} \int \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\boldsymbol{\nabla} \times \boldsymbol{J}) - \boldsymbol{J} \cdot \left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \right] d\tau'$

- Use, $\boldsymbol{\nabla} \times \boldsymbol{J} = 0$

- Hence, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = -\frac{\mu_0}{4\pi} \int \left[\boldsymbol{J} \cdot \left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \right] d\tau'$

- Now use another vector calculus identity: if f is a scalar and $\boldsymbol{A}$ is a vector, then $\boldsymbol{\nabla} \times (f\boldsymbol{A}) = f(\boldsymbol{\nabla} \times \boldsymbol{A}) + (\boldsymbol{\nabla} f) \times \boldsymbol{A}$ (prove it)

- Thus, $\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \frac{1}{\mathscr{r}^2}(\boldsymbol{\nabla} \times \hat{\boldsymbol{\mathscr{r}}}) + \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}^2}\right) \times \hat{\boldsymbol{\mathscr{r}}}$

Module 2 (Electromagnetic Theory)

Magnetostatics

- **Derivation of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$**
- The Biot-Savart law for a volume current: $\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$
- Take divergence: $\boldsymbol{\nabla} \cdot \boldsymbol{B} = -\frac{\mu_0}{4\pi} \int \left[\boldsymbol{J} \cdot \left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \right] d\tau'$ (Using, $\boldsymbol{\nabla} \times \boldsymbol{J} = 0$)
- Using a vector calculus identity: $\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \frac{1}{\mathscr{r}^2} (\boldsymbol{\nabla} \times \hat{\boldsymbol{\mathscr{r}}}) + \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}^2}\right) \times \hat{\boldsymbol{\mathscr{r}}}$
- Now, calculate $(\boldsymbol{\nabla} \times \hat{\boldsymbol{\mathscr{r}}})$ and $\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}^2}\right)$: $\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}^2}\right) = -\frac{2\,\boldsymbol{\mathscr{r}}}{\mathscr{r}^4} = -\frac{2}{\mathscr{r}^3} \hat{\boldsymbol{\mathscr{r}}}$ and $\boldsymbol{\nabla} \times \hat{\boldsymbol{\mathscr{r}}} = 0$
- Hence, $\boldsymbol{\nabla} \times \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}^2}\right) \times \hat{\boldsymbol{\mathscr{r}}} = -\frac{2}{\mathscr{r}^3} (\hat{\boldsymbol{\mathscr{r}}} \times \hat{\boldsymbol{\mathscr{r}}}) = 0$
- Therefore, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ [Divergence of the magnetic field is zero]

Module 2 (Electromagnetic Theory)

Magnetostatics

- Comparison of Magnetostatics and Electrostatics

- Divergence and Curl of Electrostatics
  - $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ (Gauss's law)
  - $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ (no name)
- These are Maxwell's equations for electrostatics

- Divergence and Curl of Magnetostatics
  - $\boldsymbol{\nabla}.\boldsymbol{B} = 0$ (no name)
  - $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ (Ampere's law)
- These are Maxwell's equations for magnetostatics

- *There are no point sources for $\boldsymbol{B}$, as compared to $\boldsymbol{E}$; there exists no magnetic analog to electric charge*.

- Hence, $\mathbf{B}$ is divergenceless ($\boldsymbol{\nabla}.\boldsymbol{B} = 0$), and there are no magnetic monopoles. It takes a **moving** electric charge to **produce** a magnetic field, and it takes another **moving** electric charge to "feel" a magnetic field.

Module 2 (Electromagnetic Theory)

Magnetostatics

- The Vector Potential

- Recall: In Electrostatics $\nabla \times \boldsymbol{E} = 0$ permitted us to introduce a scalar potential (V): $\boldsymbol{E} = -\nabla V$

- Similarly, $\nabla \cdot \boldsymbol{B} = 0$ invites the introduction of a **vector** potential $\boldsymbol{A}$ in magnetostatics: $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

- Since the *divergence of a curl is always zero*, the potential formulation automatically takes care of $\nabla \cdot \boldsymbol{B} = 0$.

- There remains the Ampere's law: $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$

- $\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ (prove it)

- Hence, $\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$

Module 2 (Electromagnetic Theory)

Magnetostatics

- The Vector Potential

- There remains the Ampere's law: $\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$.

- It can be shown that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{0}$.

- The electric potential (V) had a built-in ambiguity: one can add to V any function whose gradient is zero (for example, any *constant*), without altering $\boldsymbol{E}$. Likewise, one can add to $\boldsymbol{A}$ any function whose curl vanishes (for example, *gradient of any scalar*), with no effect on $\boldsymbol{B}$. Exploiting this property, we can have, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{0}$

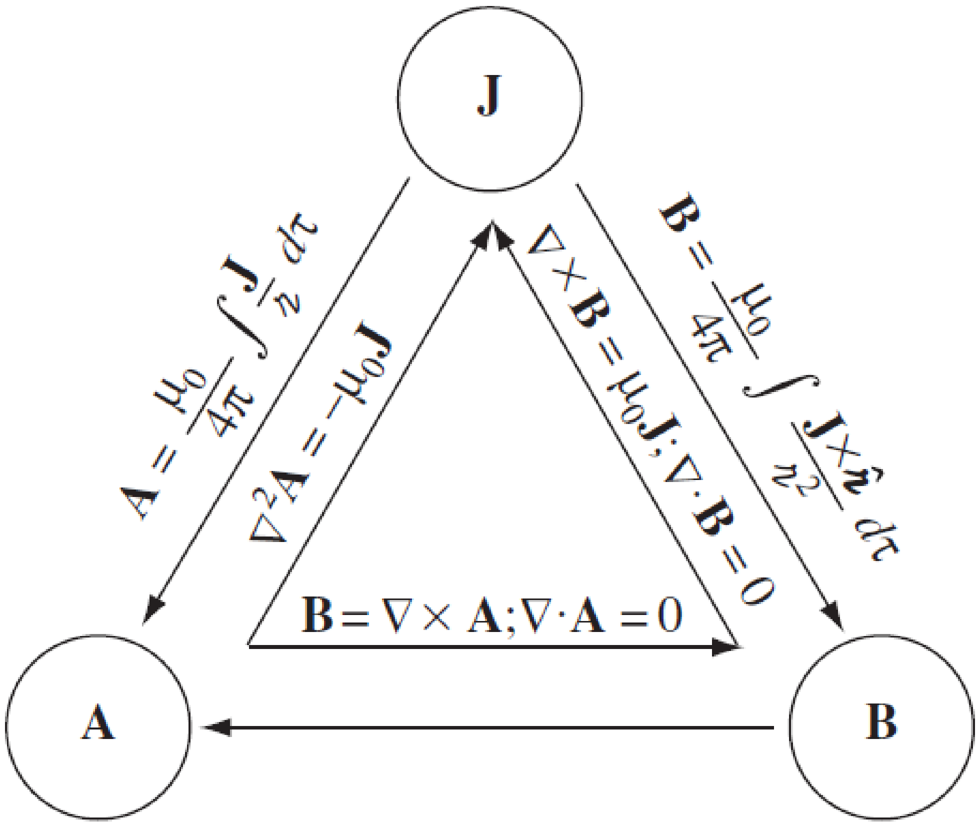
- Hence $\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$ (This is the Poisson's equation)

- Assuming $\boldsymbol{J}$ goes to zero at infinity, we can read off the solution: $\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}')}{\mathscr{r}} \, d\tau'$

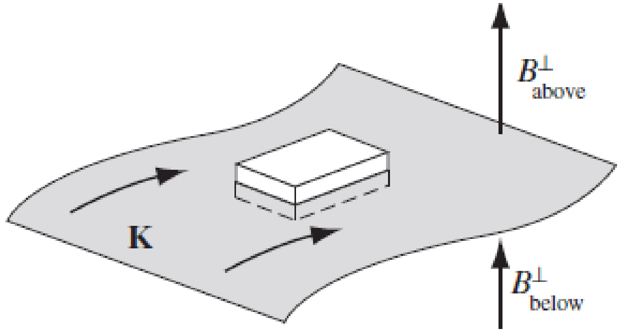
Magnetostatics

❑ Boundary Condition

❑ Like the electrostatics, a triangular diagram can be drawn to summarize the relations among the three fundamental quantities of magnetostatics, relating the current density $\boldsymbol{J}$, the field $\boldsymbol{B}$, and the potential $\boldsymbol{A}$. There is one "missing link" in the diagram: the equation for $\boldsymbol{A}$ in terms of $\boldsymbol{B}$. (See Probs. 52 and 53 from Griffith book, chapter: magnetostatics).



Module 2 (Electromagnetic Theory)



Magnetostatics

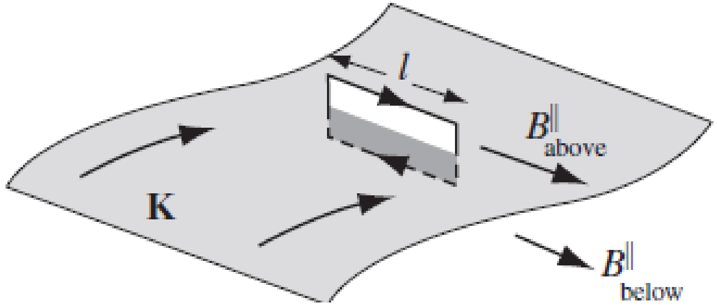
- **Boundary Condition**
- (Recall the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{D}$)
- Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current. Only this time it is the tangential component that changes.
- Boundary condition for $\boldsymbol{B}$: (a) Normal Component
- We start with $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \quad \Rightarrow \oint_S \boldsymbol{B}\,.\,d\boldsymbol{a}=0$ for a wafer-thin pillbox straddling the surface
- As the sides do not contribute ($h \to 0$), $\oint_S \boldsymbol{B}.d\boldsymbol{a} = \oint_{upper} \boldsymbol{B}.d\boldsymbol{a} + \oint_{lower} \boldsymbol{B}.d\boldsymbol{a}$

$$\Rightarrow\ B_{\perp}^{(1)}\,\Delta a + B_{\perp}^{(2)}\,(-\Delta a) = 0 \ \Rightarrow\ B_{\perp}^{(1)} - B_{\perp}^{(2)} = 0$$

- Conclusion: Normal component of $\boldsymbol{B}$ is continuous.

Module 2 (Electromagnetic Theory)

Magnetostatics



- Boundary Condition
- Boundary condition for $\boldsymbol{B}$: (a) Tangential Component
- For the tangential components, an Amperian loop running perpendicular to the current yields $\oint_C \boldsymbol{B}\,.\,d\boldsymbol{l} = \mu_0\, I_{enc}$
- As, $h \rightarrow \mathbf{0}$, only the longer sides contribute (shorter sides of length do not contribute)
- $B_{\|}^{(1)}\,(-l) + B_{\|}^{(2)}\,(l) = -\mu_0\, Kl$ (since $\int \boldsymbol{K}.\,d\boldsymbol{l} = I$)
- $B_{\|}^{(1)} - B_{\|}^{(2)} = \mu_0\, K$ (direction of $\boldsymbol{K}$ is into the page)
- Thus, the component of $\boldsymbol{B}$ that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0\, K$. **Conclusion**: Tangential component of $\boldsymbol{B}$ is discontinuous.

Module 2 (Electromagnetic Theory)

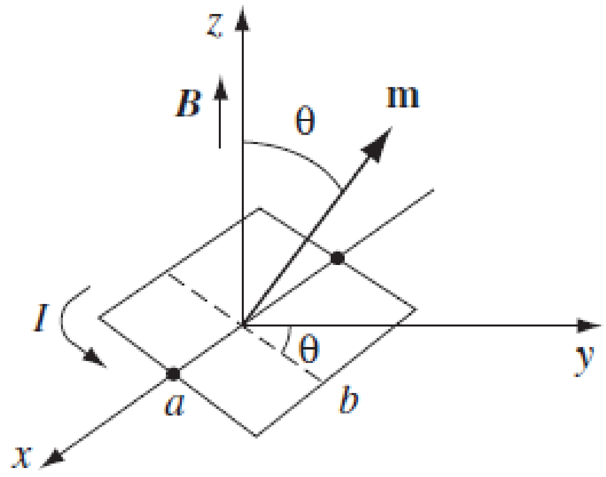
Magnetostatics

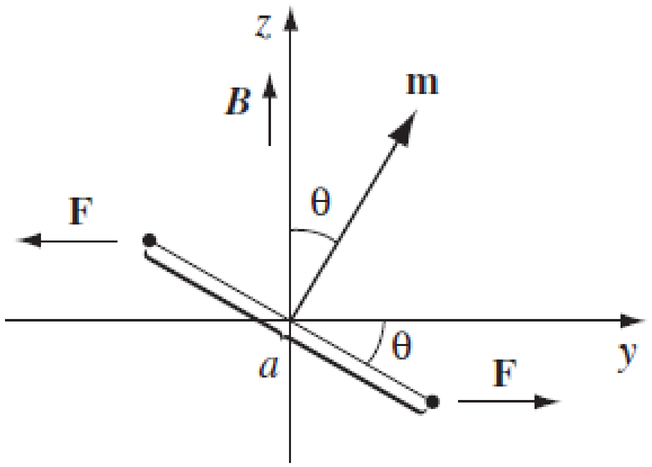
- Magnetic Fields in Matter

- A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field.

- Let's calculate the torque on a rectangular current loop (see the figures) in a uniform field $\boldsymbol{B}$.

- Consider a current carrying loop C whose plane makes an angle θ with x-y plane and the plane also passes through the origin. A magnetic field $\boldsymbol{B}$ is applied along the z-axis.

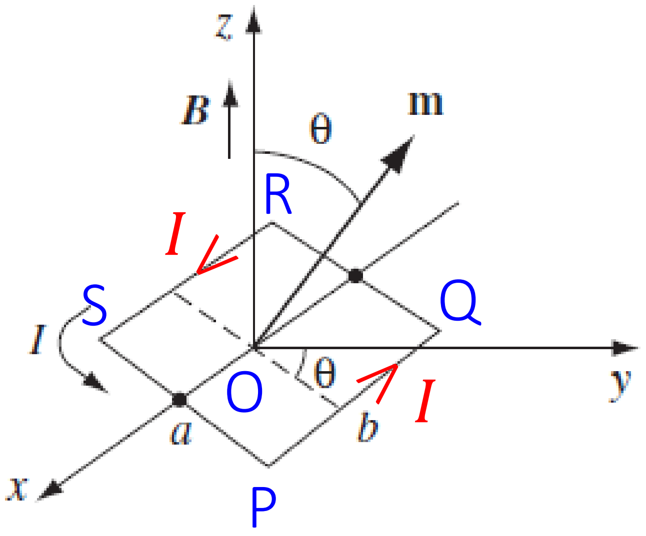




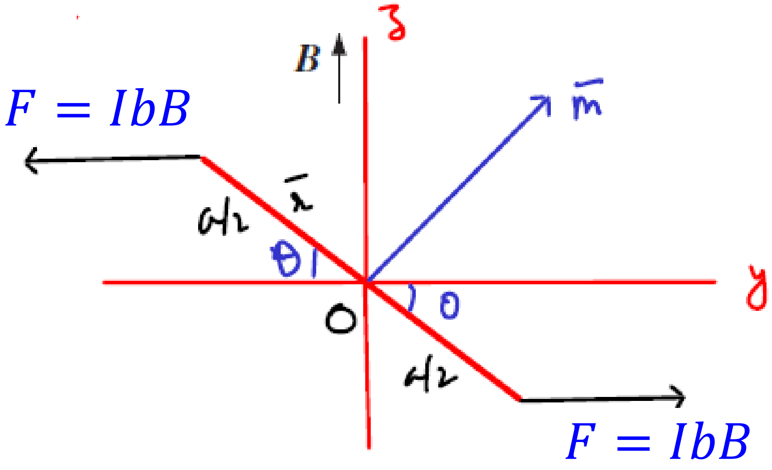
Module 2 (Electromagnetic Theory)

Magnetostatics

- Magnetic Fields in Matter
  - Force acting on QR: $\boldsymbol{F}_{QR} = I\,a\,B\sin\left(\frac{\pi}{2} - \theta\right)(-\hat{\boldsymbol{x}})$
  - Force acting on PS: $\boldsymbol{F}_{PS} = I\,a\,B\sin\left(\frac{\pi}{2} - \theta\right)\hat{\boldsymbol{x}}$



- The forces on the two sloping sides cancel (they tend to stretch the loop, but they don't rotate it).
  - Force acting on PQ: $\boldsymbol{F}_{PQ} = I\,b\,B\,\hat{\boldsymbol{y}}$
  - Force acting on RS: $\boldsymbol{F}_{RS} = I\,b\,B\,(-\hat{\boldsymbol{y}})$

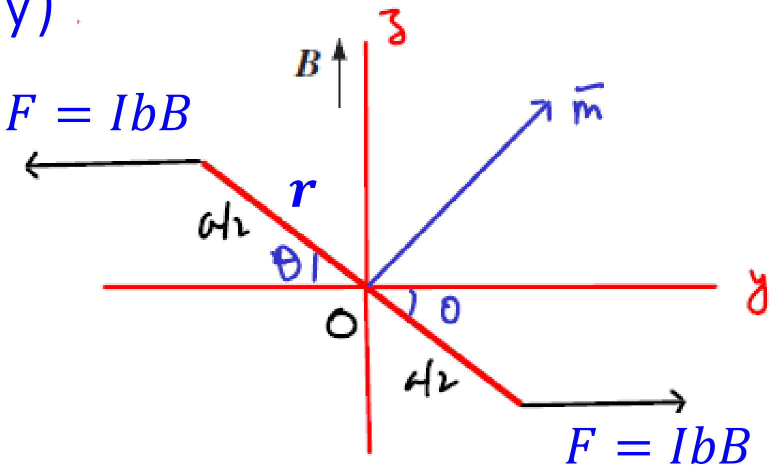


- The forces on the **horizontal** sides are likewise equal and opposite (so the net force on the loop is zero), but they generate a **torque**.

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Magnetic Fields in Matter



❑ Assume loop to be perfectly rigid. Total torque produced on the loop: $\boldsymbol{N} = 2\,\boldsymbol{r} \times \boldsymbol{F} = 2\left(\frac{a}{2}\right) F\, sin\theta\, \hat{\boldsymbol{x}} = a\, F\, sin\theta\, \hat{\boldsymbol{x}}$

❑ Using $F = IbB$, we have $\boldsymbol{N} = a\,(IbB) sin\theta\, \hat{\boldsymbol{x}} = I\,(ab)\, Bsin\theta\, \hat{\boldsymbol{x}}$

❑ Or, $\boldsymbol{N} = IABsin\theta\, \hat{\boldsymbol{x}}$ (where $A = ab$ = area of the loop)

❑ Define $m = IA$ is the magnetic dipole moment of the loop, hence, $\boldsymbol{N} = mBsin\theta\, \hat{\boldsymbol{x}}$

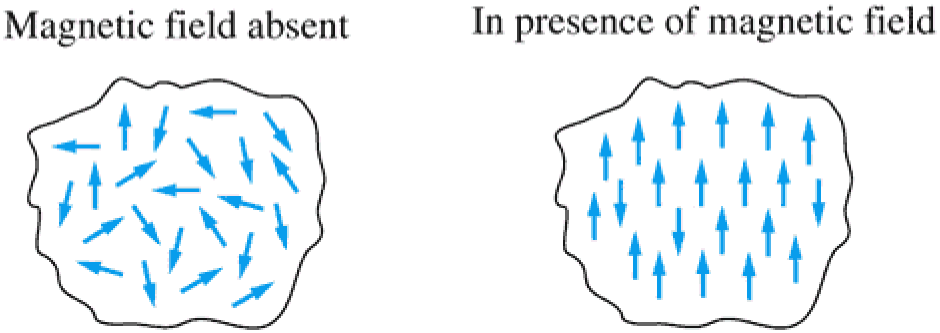
❑ So, we have, $\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$ (This torque tends to align $\boldsymbol{m}$ in the direction of $\boldsymbol{B}$)

❑ Notice that the relation $\boldsymbol{N} = \boldsymbol{m} \times \boldsymbol{B}$ is identical in form to the electrical analog: $\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$.

Module 2 (Electromagnetic Theory)

Magnetostatics

- Magnetic Fields in Matter
- A current carrying loop gives rise to a magnetic dipole moment, defined by the current flowing through the loop multiplied by the area of the loop and the direction is perpendicular to the plane of the loop.
- Presence of external magnetic field produces a torque that tries to align the dipole moment parallel to the field.
- Consider an atom. The electron revolves in an orbit, which can be viewed as a current carrying loop. Each atom produces a dipole moment.
- In presence of a strong enough magnetic field, the magnetic dipoles are aligned, and the change is opposite to the direction of the magnetic field.



Paramagnetism

Module 2 (Electromagnetic Theory)

Magnetostatics

- Magnetic Fields in Matter
- In presence of a magnetic field, matter becomes **magnetized**; that is, it contains many tiny dipoles, with a net alignment along some direction.
- The state of magnetic polarization is described by a vector quantity, $\boldsymbol{M} \equiv$ *magnetic dipole moment per unit volume*.
- $\boldsymbol{M}$ is called the magnetization; it plays a role analogous to the polarization ($\boldsymbol{P}$) in electrostatics.
- Just as electric field can be represented by $\boldsymbol{E} = -\boldsymbol{\nabla} V$ ($\because \boldsymbol{\nabla} \times \boldsymbol{E} = 0$), magnetic field can be represented by $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ ($\because \boldsymbol{\nabla}.\boldsymbol{B} = 0$). Here $\boldsymbol{A}$ is the vector potential.
- Using $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0 = \boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{a})$
  - $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ => $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ => $\boldsymbol{E} = \boldsymbol{\nabla} f = -\boldsymbol{\nabla} V$
  - $\boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{a}) = 0$ => $\boldsymbol{\nabla}.\boldsymbol{B} = 0$ => $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{A}$

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Magnetic Fields in Matter

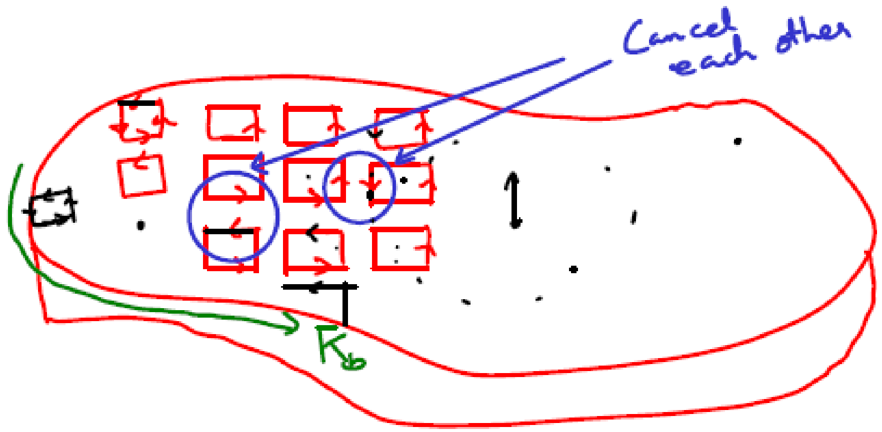
❑ Magnetic field can be represented by $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ ($\because$ $\boldsymbol{\nabla} . \boldsymbol{B} = 0$). Here $\boldsymbol{A}$ is the vector potential.

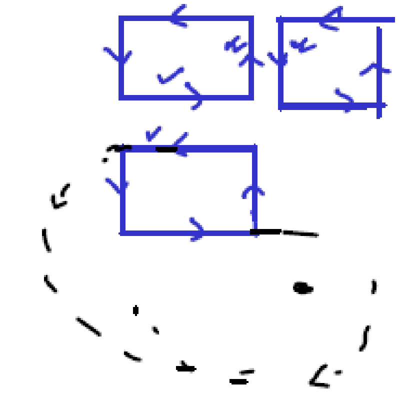
❑ The vector potential produced at a point is the same as that produced by a bound volume current density $\boldsymbol{J}_b$ and a bound surface current density $\boldsymbol{K}_b$.

❑ It can be shown that

❑ $\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M}$

❑ $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$





❑ Since the current in any arm of a loop in the bulk gets cancelled by the current in the arm of an adjacent loop, only the currents of the outermost loops do not cancel and give rise to $\boldsymbol{K}_b$.

Magnetostatics

Magnetic Fields in Matter

- Suppose magnetization is not constant throughout the sample. Then only partial cancellation will take place. So, there will be a non-zero current in the bulk also. This gives rise to $\boldsymbol{J}_b$.

- The effect of magnetization is to establish bound currents $\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M}$ within the material and $\boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$ on the surface. The field due to magnetization of the medium is just the field produced by these bound currents.

- Hence the total current can be represented as $\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f$ ($\boldsymbol{J}_f$ is the free current, for example in a conductor).

- Using Ampere's law, $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} = \mu_0(\boldsymbol{J}_b + \boldsymbol{J}_f)$

$$\Rightarrow \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{J}_b + \boldsymbol{J}_f = \boldsymbol{\nabla} \times \boldsymbol{M} + \boldsymbol{J}_f$$

Module 2 (Electromagnetic Theory)

Magnetostatics

- Magnetic Fields in Matter

- $\frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) = \nabla \times \boldsymbol{M} + \boldsymbol{J}_f$

- This can be written as: $\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_f$

- Let, $\left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) \equiv \boldsymbol{H}$ is the magnetic field strength (in A/m)

This gives, (a) $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$

- In integral form, $\oint_S (\nabla \times \boldsymbol{H}).d\boldsymbol{a} = \oint_S \boldsymbol{J}_f\,.d\boldsymbol{a} = I_{f,enc}$

where $I_{f,enc}$ is the total *free* current passing through the Amperian loop.

- Therefore, $\oint_C \boldsymbol{H}\,.d\boldsymbol{l} = I_{f,enc}$

- (b) Divergence of $\boldsymbol{H}$ gives: $\nabla\,.\boldsymbol{H} = \nabla\,.\left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = -\nabla\,.\boldsymbol{M} \quad (\because \nabla.\boldsymbol{B} = 0)$

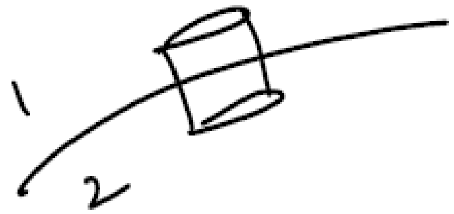
- $\boldsymbol{H}$ plays a role in magnetostatics analogous to $\boldsymbol{D}$ in electrostatics.

Magnetostatics

❑ Boundary conditions

❑ The magnetostatic boundary conditions can be given in terms of $\boldsymbol{H}$ and the free current

(a) Recall $\boldsymbol{\nabla} . \boldsymbol{H} = -\boldsymbol{\nabla} . \boldsymbol{M} \quad \Rightarrow \quad \boldsymbol{\nabla} . (\boldsymbol{H} + \boldsymbol{M}) = 0$



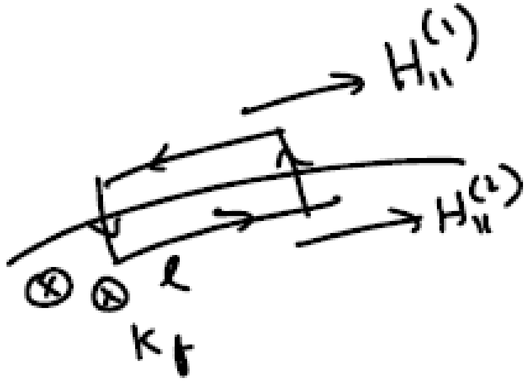
❑ $(H+M)_\perp^{(1)} = (H+M)_\perp^{(2)} \quad \Rightarrow \quad H_\perp^{(1)} + M_\perp^{(1)} = H_\perp^{(2)} + M_\perp^{(2)}$

❑ Thus, $H_\perp^{(1)} - H_\perp^{(2)} = M_\perp^{(2)} - M_\perp^{(1)}$ $\quad \rightarrow H_\perp$ is discontinuous

$\boldsymbol{\nabla} . (\boldsymbol{H} + \boldsymbol{M}) = 0 \ \rightarrow \boldsymbol{\nabla} . \boldsymbol{f} = 0$

This will give, $f_\perp^{(1)} = f_\perp^{(2)}$

(b) Recall $\oint_C \boldsymbol{H} . d\boldsymbol{l} = I_{f,enc}$



❑ $H_\parallel^{(1)}(-l) + H_\parallel^{(2)}(l) = -K_f l$

❑ $H_\parallel^{(1)} - H_\parallel^{(2)} = K_f$ $\quad \rightarrow H_\parallel$ is discontinuous

❑ If $K_f = 0$, then $H_\parallel^{(1)} = H_\parallel^{(2)}$. In general, both $H_\perp$ and $H_\parallel$ components are discontinuous.

Magnetostatics

- ❑ Boundary conditions
- ❑ Summary
- ❑ $\boldsymbol{H} = \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right)$
- ❑ $\boldsymbol{\nabla} \,.\, \boldsymbol{H} = -\boldsymbol{\nabla} \,.\, \boldsymbol{M}$ => $H_{\perp}^{(1)} - H_{\perp}^{(2)} = M_{\perp}^{(2)} - M_{\perp}^{(1)}$
- ❑ $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f$ => $H_{\parallel}^{(1)} - H_{\parallel}^{(2)} = K_f$

Module 2 (Electromagnetic Theory)

- Maxwell's equations for static field [here, $E \neq E(t), B \neq B(t)$]

General

- $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$
- $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$
- $\boldsymbol{\nabla}.\boldsymbol{B} = 0$
- $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$

Inside Matter

- $\boldsymbol{\nabla}.\boldsymbol{D} = \rho_f$
- $\boldsymbol{\nabla} \times \boldsymbol{D} = \boldsymbol{\nabla} \times \boldsymbol{P}$
- $\boldsymbol{\nabla}.\boldsymbol{H} = -\boldsymbol{\nabla}.\boldsymbol{M}$
- $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f$

Boundary conditions: General

- $E_\perp^{(1)} - E_\perp^{(2)} = \frac{\sigma}{\varepsilon_0}$
- $E_\parallel^{(1)} - E_\parallel^{(2)} = 0$
- $B_\perp^{(1)} - B_\perp^{(2)} = 0$
- $B_\parallel^{(1)} - B_\parallel^{(2)} = \mu_0 K_f$

Boundary conditions: Inside Matter

- $D_\perp^{(1)} - D_\perp^{(2)} = \sigma_f$
- $D_\parallel^{(1)} - D_\parallel^{(2)} = P_\parallel^{(1)} - P_\parallel^{(2)}$
- $H_\perp^{(1)} - H_\perp^{(2)} = M_\perp^{(2)} - M_\perp^{(1)}$
- $H_\parallel^{(1)} - H_\parallel^{(2)} = K_f$

Module 2

Date: 04.10.2023

Lecture: 5

PH113: Physics (B.Tech. I)

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

Text Book: Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press (Seventh Edition, 2018)

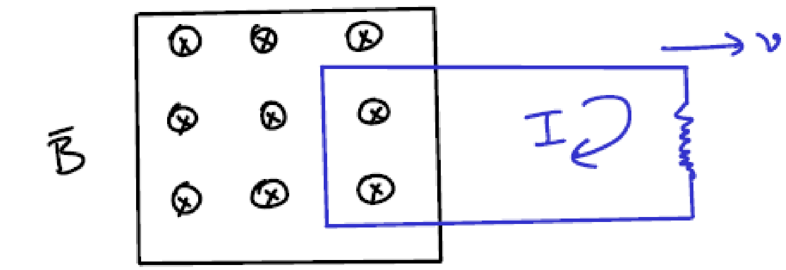
Reference Book: David J. Griffiths, Introduction to Electrodynamics, Pearson (Fourth Edition, 2014)

Class structure: 4 Lectures including 1 Tutorial per week. (8 hours ~ 2 weeks for this module!)

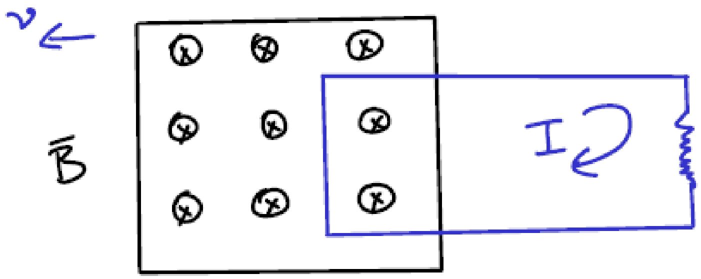
Module 2 (Electromagnetic Theory)

Magnetostatics

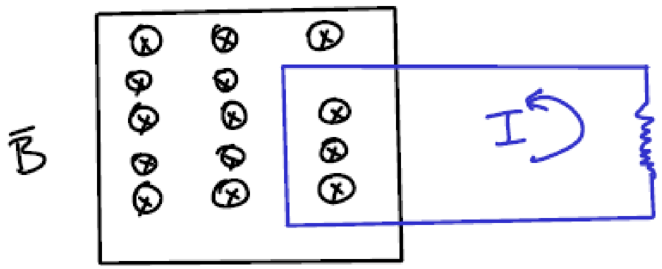
❑ Faraday's law (Experiment done by Faraday in 1831)



The loop was pulled rightward with velocity $\boldsymbol{v}$. Current I flowed clockwise.



Magnet was pulled leftward with velocity $\boldsymbol{v}$ holding the loop still. Again, current I flowed clockwise.

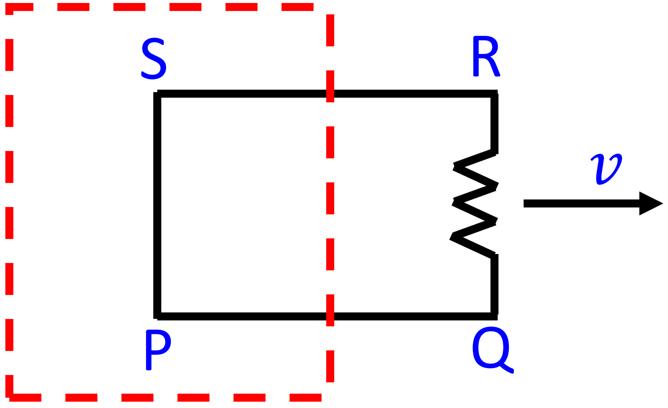


Magnetic field strength was changed (increased the field). Current I was anticlockwise.

Module 2 (Electromagnetic Theory)

Magnetostatics

- Faraday's law
- Lenz's law: Nature hates a change in flux. The three cases in previous slide are special cases of Lenz's law.
- Faraday deduced: A changing magnetic field induces an electric field.
- Electromotive force: $V_{emf} \equiv$ amount of work done by a unit charge for making a complete round of the loop.
- $V_{emf} = \oint \boldsymbol{f} \, . \boldsymbol{dl}$ (where, $\boldsymbol{f}$ is the force experienced by the charge)

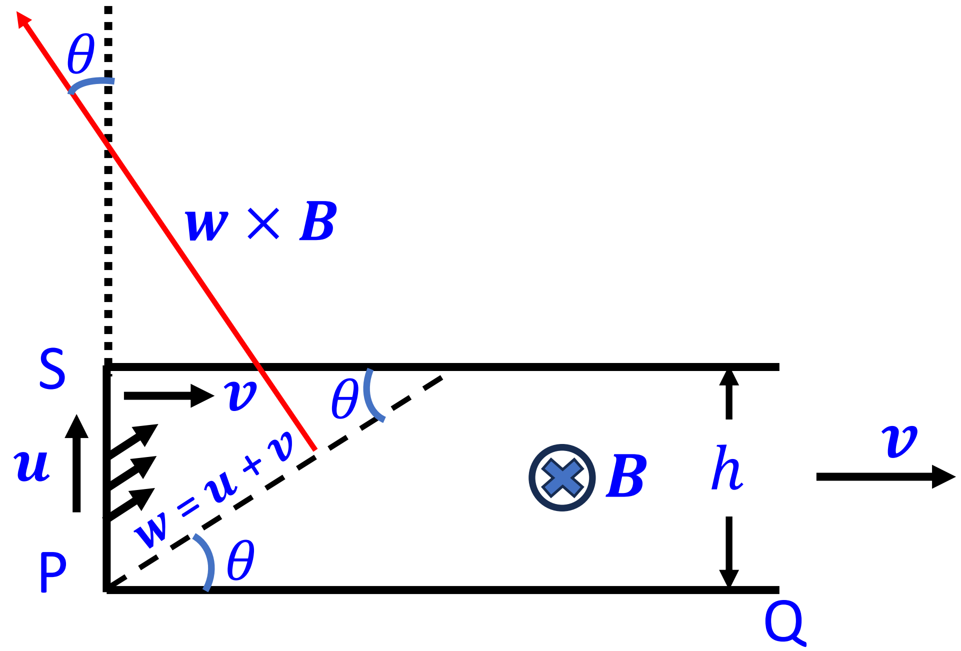


- Arms SR and PQ do not contribute ($\because \boldsymbol{u} \times \boldsymbol{B}$ is perpendicular to $\boldsymbol{dl}$ where $\boldsymbol{u}$ is the velocity of charge)
- Side RQ is outside the field region, so it does not contribute. We need to consider only the side PS.

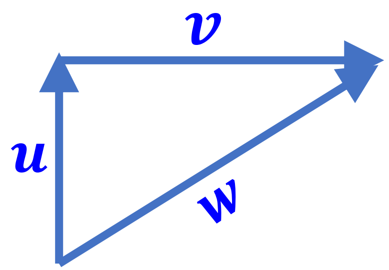
Module 2 (Electromagnetic Theory)

Magnetostatics

- Faraday's law



- $V_{emf} = \oint \boldsymbol{f} . \boldsymbol{dl}$ (where, $\boldsymbol{f}$ is the force experienced by the charge)
- $V_{emf} = \int_P^S (\boldsymbol{w} \times \boldsymbol{B}). \boldsymbol{dl}$
- Here $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is the resultant velocity of the charge due to its original motion (with velocity $\boldsymbol{u}$) and the horizontal velocity (velocity $\boldsymbol{v}$ of pulling). So, $(\boldsymbol{w} \times \boldsymbol{B})$ makes an angle θ with the horizontal.

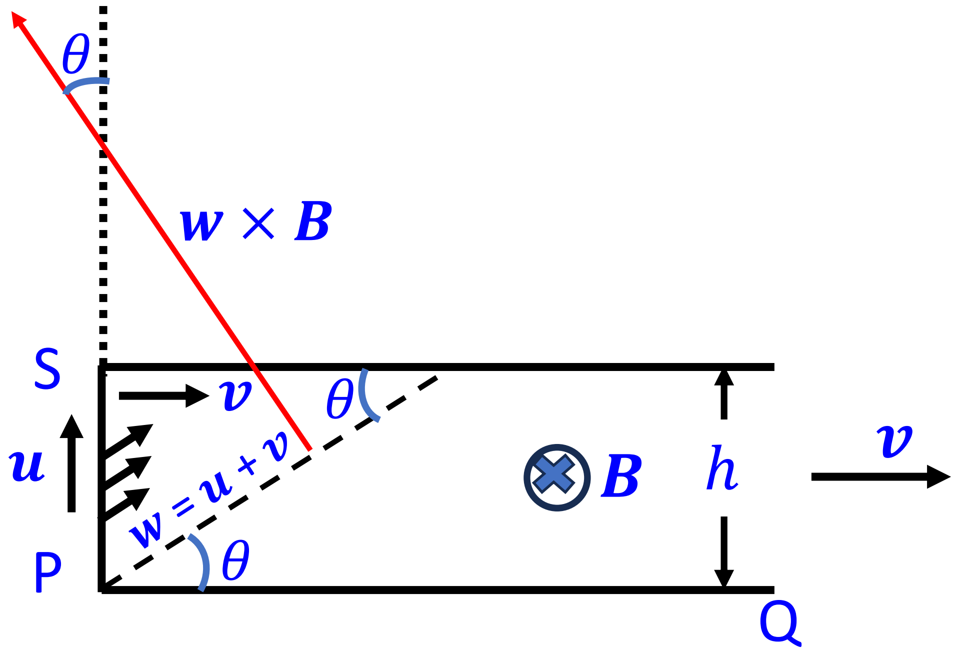


- Hence, $V_{emf} = \int_P^S (\boldsymbol{w} \times \boldsymbol{B}). \boldsymbol{dl} = \int_P^S |\boldsymbol{w} \times \boldsymbol{B}|\, dl\, cos\theta$

Module 2 (Electromagnetic Theory)

Magnetostatics

❑ Faraday's law



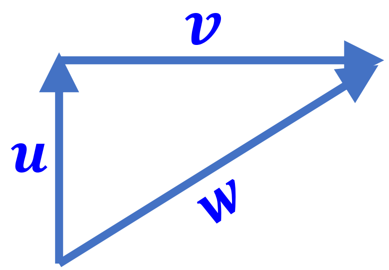
❑ $V_{emf} = \int_P^S (\boldsymbol{w} \times \boldsymbol{B}).\boldsymbol{dl} = \int_P^S |\boldsymbol{w} \times \boldsymbol{B}|\, dl\, cos\theta$

$$= \int_P^S w\, B\, dl\, cos\theta$$

$$= B \int_P^S (v\, sec\theta)\, dl\, cos\theta$$

(since, from the vector triangle, $v = w\, cos\theta$)

❑ Hence, $V_{emf} = vB \int_P^S dl = vBh$

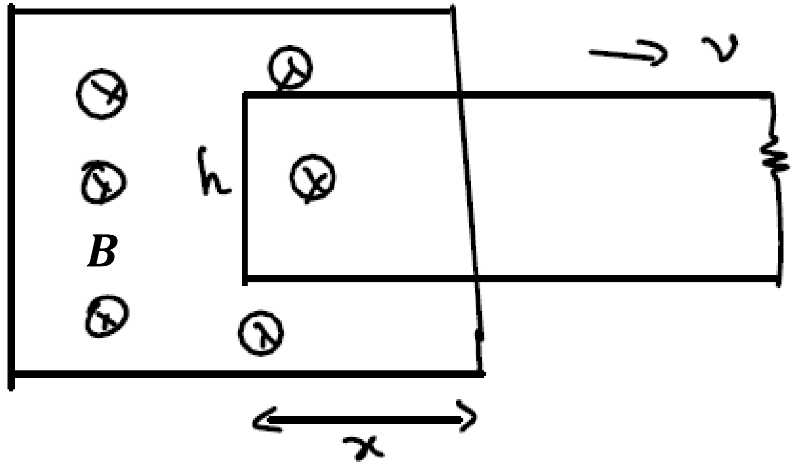


❑ Now consider the rate of flux in the loop. Faraday found experimentally: $V_{emf} = -\frac{d\Phi_B}{dt}$ (where Φ_B is the magnetic flux through the loop).

Module 2 (Electromagnetic Theory)

Magnetostatics

- Faraday's law



- Magnetic flux through the loop: $\Phi_B = \int \boldsymbol{B} \, . \, \boldsymbol{da} = Bhx$
- $d\Phi_B = -Bh\,dx$ (since, x decreases with time)
- Hence, $\frac{d\Phi_B}{dt} = -B\,h\frac{dx}{dt} = -B\,h\,v$
- Hence, $V_{emf} = -\frac{d\Phi_B}{dt} = Bhv$
- Faraday deduced that the current was caused by an "induced" electric field.
- If this induced field is $\boldsymbol{E}$, then $V_{emf} = \oint \boldsymbol{E} \, . \, \boldsymbol{dl} = -\frac{d\Phi_B}{dt}$

Module 2 (Electromagnetic Theory)

Magnetostatics

- Faraday's law

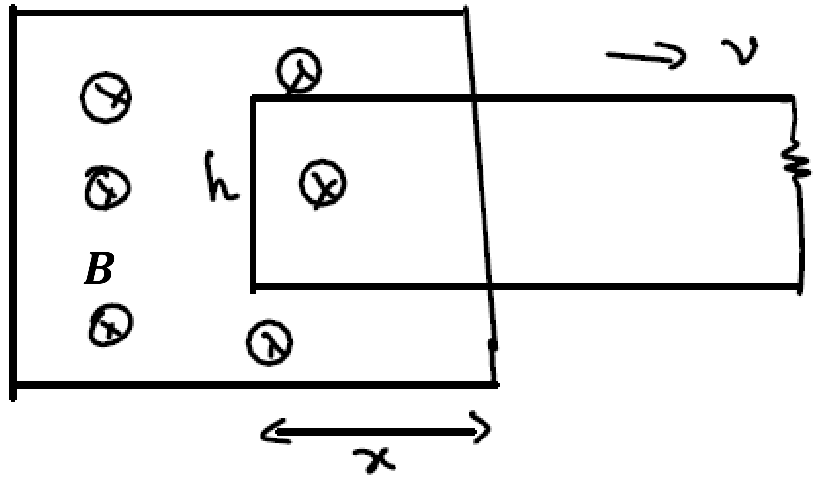
- If the induced electric field is $\boldsymbol{E}$, then $V_{emf} = \oint \boldsymbol{E} \, . \, \boldsymbol{dl} = -\frac{d\Phi_B}{dt}$

- Hence, $\oint \boldsymbol{E} \, . \, \boldsymbol{dl} = -\frac{d}{dt} \int \boldsymbol{B} \, . \, \boldsymbol{da} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \, . \, \boldsymbol{da}$ (use $\Phi_B = \int \boldsymbol{B} \, . \, \boldsymbol{da}$)

- Now, use Stokes' theorem:

$$\int (\nabla \times \boldsymbol{E}) \, . \, \boldsymbol{da} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \, . \, \boldsymbol{da}$$

- By comparison, $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$



Module 2 (Electromagnetic Theory)

Magnetostatics

- Faraday's law

- If there is a change in magnetic flux, we have an induced electric field is $\boldsymbol{E}$, such that

$$V_{emf} = \oint \boldsymbol{E} \,.\, \boldsymbol{dl} = -\frac{d\Phi_B}{dt} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \,.\, \boldsymbol{da}$$

- Since, $\oint \boldsymbol{E} \,.\, \boldsymbol{dl} = \int (\boldsymbol{\nabla} \times \boldsymbol{E}) \,.\, \boldsymbol{da}$, we have, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \neq 0$

- This modifies the Maxwell's equation $\boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}$ for static field.

- Faraday's law generalizes the electrostatic rule $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$ to the time-dependent regime.

Module 2 (Electromagnetic Theory)

Magnetostatics

- ❑ Displacement Current
- ❑ Another equation for the static field is $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$.
- ❑ This equation also gets modified if the fields are time-dependent.

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$$\boldsymbol{\nabla}.\, (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0\, (\boldsymbol{\nabla} \,.\, \boldsymbol{J})$$

Divergence of a curl is **always** zero. Hence, LHS of this equation is zero due to vector identity [$\boldsymbol{\nabla}.\, (\boldsymbol{\nabla} \times \boldsymbol{B}) = 0$]. But, there is no reason why $\boldsymbol{\nabla} \,.\, \boldsymbol{J} = \boldsymbol{0}$ so RHS ≠ 0 (in general)

- ❑ We know from the equation of continuity: $\boldsymbol{\nabla} \,.\, \boldsymbol{J} = -\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}}$ (statement of conservation of charge)
- ❑ But, $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ $\quad\Rightarrow\quad \rho = \varepsilon_0(\boldsymbol{\nabla}.\boldsymbol{E})$

Magnetostatics

❑ Displacement Current

❑ $\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ $\quad\Rightarrow\quad \rho = \varepsilon_0(\boldsymbol{\nabla}.\boldsymbol{E})$

❑ Hence from the Eqn. of continuity, $\boldsymbol{\nabla}\,.\,\boldsymbol{J} = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\boldsymbol{\nabla}.\boldsymbol{E}) = -\boldsymbol{\nabla}\,.(\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t})$

❑ This gives, $\boldsymbol{\nabla}\,.\,(\boldsymbol{J} + \varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t}) = 0$

$$\Rightarrow \boldsymbol{\nabla}\,.\boldsymbol{J}_1 = 0$$ (Consider, $\boldsymbol{J} + \varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}_1$)

❑ Now, $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0\,\boldsymbol{J}_1$ $\quad\Rightarrow \boldsymbol{\nabla}.\,(\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0\,(\boldsymbol{\nabla}\,.\,\boldsymbol{J}_1) = 0$

(the anomaly has disappeared once $\boldsymbol{J}$ is replaced by $\boldsymbol{J}_1$)

Magnetostatics

❑ Displacement Current

❑ So, we have the modified form of Maxwell's equation if the fields are **not** static:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \, \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

❑ The quantity $\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \equiv \boldsymbol{J}_d$ is called the displacement current.

Magnetostatics

❑ Maxwell's Equations

❑ The final (general) form of Maxwell's equations are:

$$\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

✓ $$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0$$

✓ $$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

❑ For static fields the terms $\frac{\partial \boldsymbol{B}}{\partial t}$ and $\frac{\partial \boldsymbol{B}}{\partial t}$ will vanish.

Module 2

Module – 2: Electromagnetic Theory

Gradient, Divergence and Curl, Statement of Gauss theorem & Stokes theorem, Gauss's law, Applications, Concept of electric potential, Relationship between E and V, Polarization of dielectrics, dielectric constant, Boundary conditions for E & D, Gauss's law in magnetostatics, Ampere's circuital law, Boundary conditions for B & H, Equation of continuity, Displacement current, Maxwell's equations. [8]

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Module 2

Next Class

Questions?

Module 2

Date: 09.10.2023

Lecture: 6 (Tutorial Class)

PH113 / Assignment - Module II: EM Theory

(Dated: September 21, 2023)

1. **Gradient, Divergence, and Curl.**

   (a) Calculate the gradient of (i) $U = x^2$, (ii) $U = r^2$, (iii) $\mathbf{c} \cdot \mathbf{r}$ ($\mathbf{c}$ is a constant vector).

   Solution. (i) $2x\hat{\mathbf{i}}$, (ii) $2\mathbf{r}$, (iii) $\mathbf{c}$.

   (b) For $U = f(r)$, show that $\nabla U = \frac{df}{dr}\left[\frac{\mathbf{r}}{r}\right]$.

   Solution. $U = f(r) = f(r(x, y, z))$

   Now, $\frac{\partial f}{\partial x} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}$

(c) Calculate the divergence of (i) $x\hat{\mathbf{i}}$, (ii) $\mathbf{r}$, (iii) $\mathbf{r}/r^3$, (iv) $r\mathbf{c}$ ($\mathbf{c}$ is a constant).

Solution. (i) 1, (ii) 3, (iii) 0, (iv) $(\mathbf{r} \cdot \mathbf{c})/r$

Comment: The divergence of a vector field represents the flux generation per unit volume at each point of the field. (Divergence because it is an efflux not an influx.)

(d) Calculate the curl of (i) $-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, (ii) $x^2y^2\hat{\mathbf{k}}$.

Solution. (i) $2\hat{\mathbf{k}}$, (ii) $2x^2y\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}$

(e) Show that $\nabla \cdot (\nabla \times \mathbf{v}) = 0$

(f) Show that $\nabla \times (\nabla f) = 0$

- Remember, ”d c g(enerator)”, which stands for div of a curl is zero, curl of a grad is zero!
- Note:
  - A vector field with zero divergence is said to be solenoidal.
  - A vector field with zero curl is said to be irrotational.
  - A scalar field with zero gradient is constant.

Q. Show that:

(1) $\nabla \cdot (\nabla \times \vec{v}) = 0$

(2) $\nabla \times (\nabla f) = 0$.

<u>Soln</u> (1) $\nabla \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0.$$

(2) Try to do yourself.

(g) Show that the curl of a $\vec{r}$ is zero. In general, derive the conditions when the curl of a radial vector field is zero.

Solution. The first part is trivially straightforward. For the second part, simply consider a vector field $\mathbf{A} = f(r)\hat{r}$, and show that $\nabla \times \mathbf{A} = 0$. If $f_x(r)$, $f_y(r)$ and $f_z(r)$ are the x,y and z components of $\mathbf{A}$, then we equate the curl to zero and obtain:

$$\frac{\partial f_z(r)}{\partial y} = \frac{\partial f_y(r)}{\partial z}$$
$$\frac{\partial f_x(r)}{\partial z} = \frac{\partial f_z(r)}{\partial x}$$
$$\frac{\partial f_y(r)}{\partial x} = \frac{\partial f_x(r)}{\partial y}$$

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(h) Calculate the Laplacian of (i) r^2, (ii) xy^2z^3, (iii) $1/r$.

Solution. (i) 6, (ii) $2xz^3 + 6xy^2z$, (iii) 0.

For (iii), we note that:

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} &= \frac{\partial}{\partial x}[-x(x^2+y^2+z^2)^{-3/2}] \\ &= -(x^2+y^2+z^2)^{-3/2} + 3\,x\,x\,(x^2+y^2+z^2)^{-5/2} \\ &= (1/r^3)(-1+3x^2/r^2)\end{aligned}$$

Therefore, upon adding y and z terms,we have :

$$\nabla^2\left(\tfrac{1}{r}\right) = \left(\tfrac{1}{r^3}\right)\left(-3+3\tfrac{(x^2+y^2+z^2)}{r^2}\right) = 0$$

2. **Maxwell's equations.**

(a) Write down the static and time-dependent Maxwell's equation in vacuum.

(b) If the electric field in some region is given by:

$E = \frac{A\hat{r} + B\sin\theta\cos\phi}{r}$,

where A and B are constants, what is the charge density?

Solution.

$$\rho = \epsilon_0\left(\frac{A}{r^2} - \frac{B}{r^2}\sin\phi\right)$$

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(c) Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ) of radius R..

For a symmetric spherical charge distribution, we can choose a surface S centered about the charge distribution. Integrating the electric field ($\boldsymbol{E}$) over this surface - the electric flux - is proportional to the enclosed charge.

This is the statement of Gauss's Law: $\Phi = \oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$ [Here, $d\boldsymbol{a}$ is a vector element of area of the surface S, that is, it is a vector of magnitude equal to the area of a differential segment of the surface and points normal to the surface at that location.]

$$\oint_S \boldsymbol{E}.d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \Rightarrow \quad E\oint_S d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \Rightarrow E\, 4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}$$

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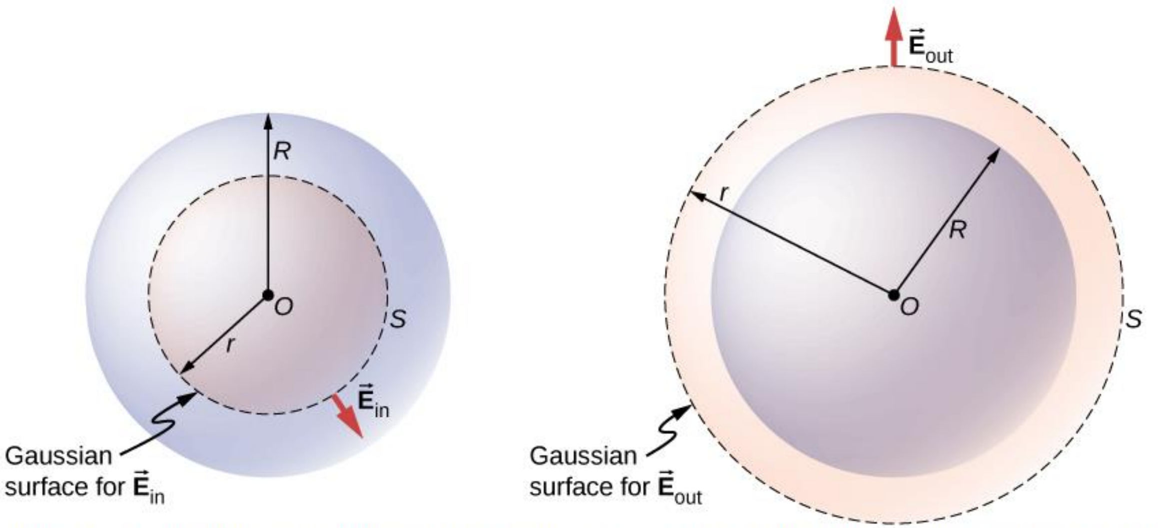
(c) Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ) of radius R.



A spherically symmetrical charge distribution and the Gaussian surface used for finding the field (a) inside and (b) outside the distribution.

If point P is located **outside** the charge distribution ($r \geq R$), then the Gaussian surface containing P encloses all charges in the sphere. In this case, Q_{enc} equals the total charge in the sphere.

On the other hand, if point P is **inside** the spherical charge distribution ($r < R$), then the Gaussian surface encloses a smaller sphere than the sphere of charge distribution. In this case, Q_{enc} is less than the total charge present in the sphere.

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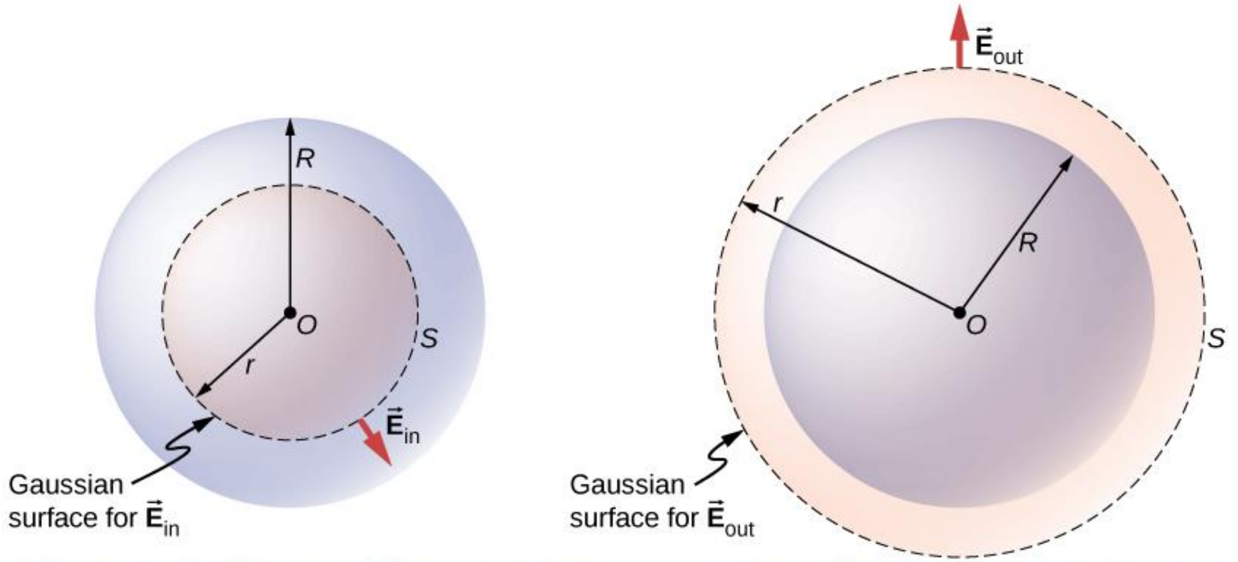
(c) Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density ρ) of radius R.

For the constant density sphere, the enclosed charge is

$$Q_{enc} = \int \rho(\boldsymbol{r})\, d\tau = \rho \int d\tau = \frac{4}{3}\pi r^3 \rho$$



A spherically symmetrical charge distribution and the Gaussian surface used for finding the field (a) inside and (b) outside the distribution.

Therefore, $E_{in} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{\frac{4}{3}\pi r^3 \rho}{4\pi\varepsilon_0 r^2} = \frac{\rho r}{3\varepsilon_0}$

Note: for outside the distribution:$Q_{enc} = Q_{total} = \int \rho(\boldsymbol{r})\, d\tau = \rho \int d\tau = \frac{4}{3}\pi R^3 \rho$

and, $E_{out} = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = \frac{Q_{total}}{4\pi\varepsilon_0 r^2} = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0 r^2} = \frac{\rho R^3}{3\varepsilon_0 r^2}$

(d) Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin: $r = kr$, for some constant k.

Solution. $$Q_{\text{enc}} = \pi k r^4$$

$$E(r) = kr^2/4\epsilon_0$$

(e) Electric field due to an infinite charge sheet with charge density σ.

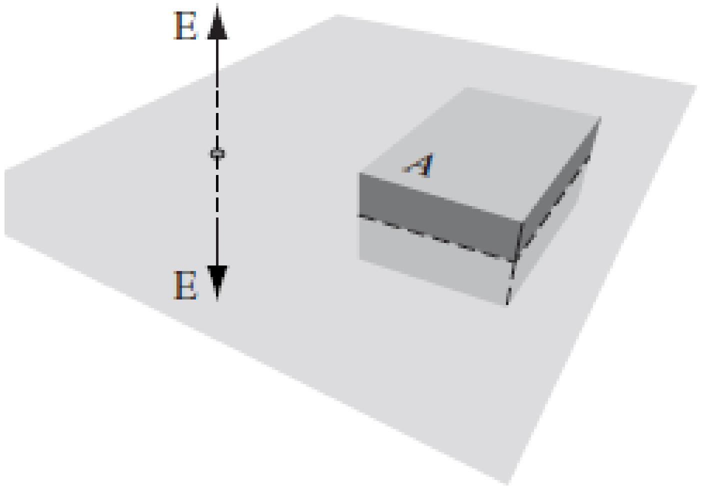
Solution

Draw a "Gaussian pillbox," extending equal distances above and below the plane (Fig. 22). Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\mathrm{enc}}.$$

In this case, $Q_{\mathrm{enc}} = \sigma A$, where A is the area of the lid of the pillbox. By symmetry, $\mathbf{E}$ points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$$



whereas the sides contribute nothing. Thus

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0}\sigma A,$$

(f) Suppose the electric potential is given by the expression $V(r) = Ae^{-kr}/r$, for all r (A and k are constants). Find the electric field $\mathbf{E}(r)$, the charge density $\rho(r)$, and the total charge Q.

Solution. The electric field $E(\mathbf{r})$ can be immediately obtained from the electric potential, using $E = \nabla V$. ρ can be found by evaluating $\nabla \cdot E$, and Q by integrating over ρ.

We get

$$\begin{aligned} E(r) &= \left(kAe^{-kr}/r + Ae^{-kr}/r^2\right)\hat{r} \\ \rho(r) &= \epsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - k^2e^{-kr}/r\right] \\ Q &= 4\pi\epsilon_0 A \end{aligned} \tag{2}$$

We used (without proof) $\nabla \cdot (\hat{r}/r^2) = \delta^3\mathbf{r}$.

3. **Determination of E for some distribution of charges.** For example,

   (a) Consider two charges q_1 and q_2 at $\pm d/2$ from the origin, along the x-axis. Obtain the electric field at (i) distance z away from the origin along z, (ii) along the distance $x < -d/2$ and $x > +d/2$ along the x axis.

   (b) $+q$ and $-q$ charges in the same geometry.

   Comment: This is the concept of an electric dipole. Note what happens for situations similar to (ii) in the problem above.

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(c) What happens is the charge q is smeared over length L along the x-axis centered at origin?

Solution. Considering charge distribution to be $\lambda = q/L$, the contribution due to a charge element is given by (see Fig. 1):

$$dE = k\frac{dq}{(x^2+y^2)} = \frac{k\lambda dx}{(x^2+y^2)}$$

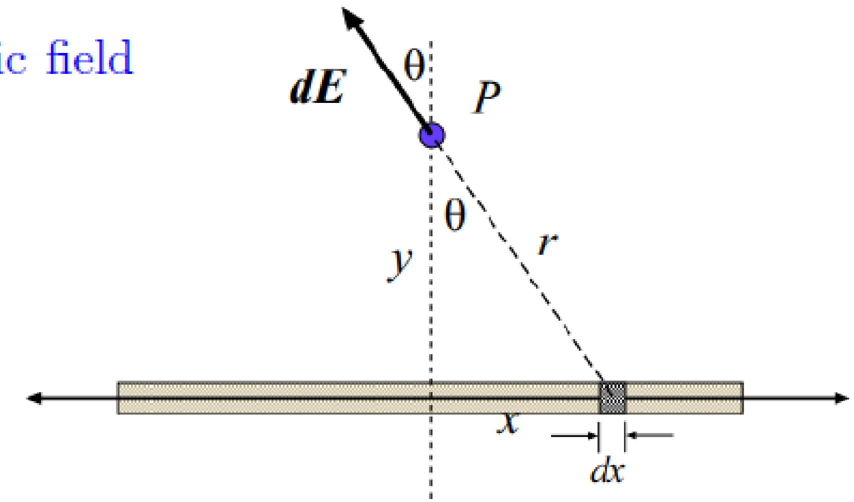
It should be clear that there will be no x component of the resultant electric field by symmetry. After carrying out the integral for E_y, we get:

$$E = \frac{q}{2\pi\epsilon_0 y}\frac{1}{(L^2+4y^2)^{1/2}}$$

(d) E at some point along the axis of symmetry for a uniform charge distributed as a ring.

Solution. Consider a ring of charge with radius R and total charge q. For a point on the axis of the ring a distance z from the center, the magnitude of the electric field (which points along the z axis) is

$$E = \frac{q^2}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$



(e) Calculate the electric field at a distance z away along the axis of a disk with uniform charge density σ. What happens when the radius of the disk tends to be infinitely large?

Solution. For a disk of radius R with charge density σ, the magnitude of the electric field at z distance away from the axis is given by:

$$E = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$

In the limit $R \to \infty$, we recover the formula for the E due to infinite charge sheet: $E = \sigma/2\epsilon_0$.

4. Consider the situation shown in the figure below. Calculate the electric field at the center of the square if $q = 1.0 \times 10^{-8}$ C and $a = 5.0$ cm.

Solution. The center of a square is equidistance from its vertices where the charges are placed. The distance between the center and the vertices is half the diagonal of square, $r = \sqrt{2}a/2 = a/\sqrt{2} = 3.55 \times 10^{-2}$m. Therefore, the magnitudes of the contribution to $\mathbf{E}$ from each of the point charges is:

$$E_{1.0q} = k\frac{q}{r^2} = 7.13 \times 10^4 \text{ N/C}$$

$$E_{2.0q} = 2E_{1.0q} = 1.43 \times 10^5 \text{ N/C}$$

The directions are shown in $\boxed{3}$. The angle between the direction of $\mathbf{E}$ and the horizontal/vertical axis is some multiple of 45°. After doing the vector sum correctly, we end up with:

$$\mathbf{E}_{\text{total}} = (0.0\,\hat{x} + 1.02 \times 10^5\,\hat{y}) \text{ N/C}$$

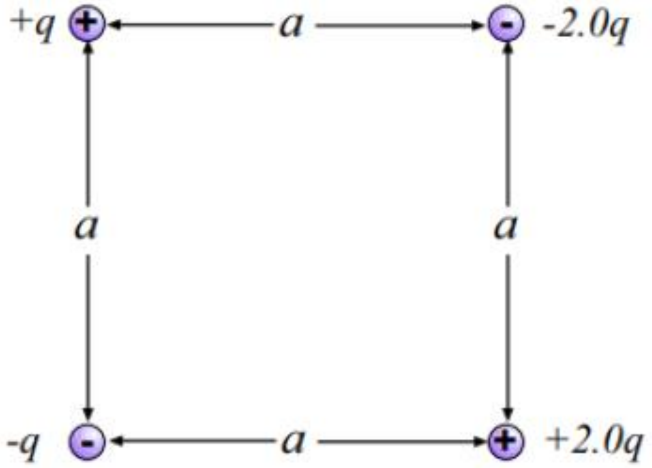


FIG. 2. Charge distribution for Q3(e).

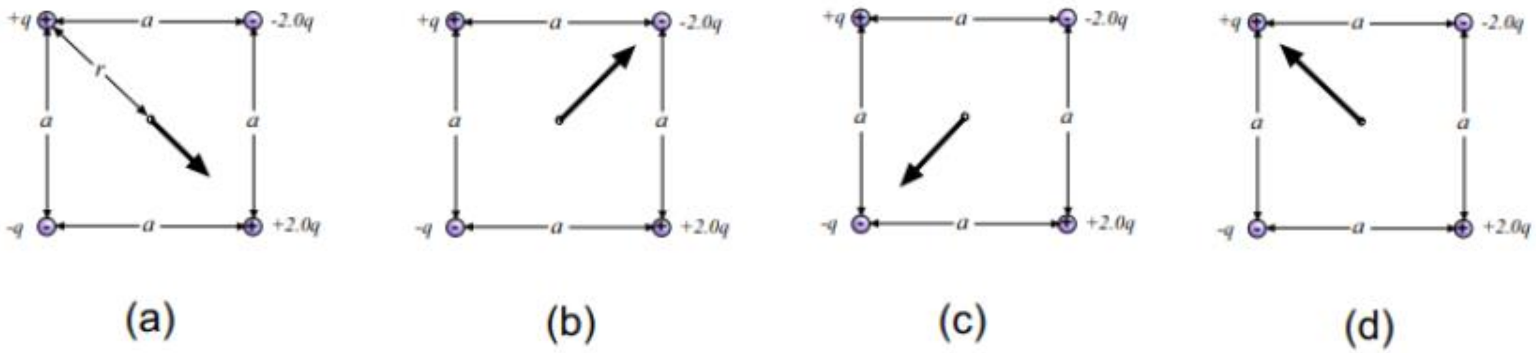


FIG. 3. Directions of **E** due to charges position on the vertices of a square.

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5. An object having a net charge of 24 μC is placed in a uniform electric field of 610 N/C directed vertically. What is the mass of this object if it "floats" in the field?

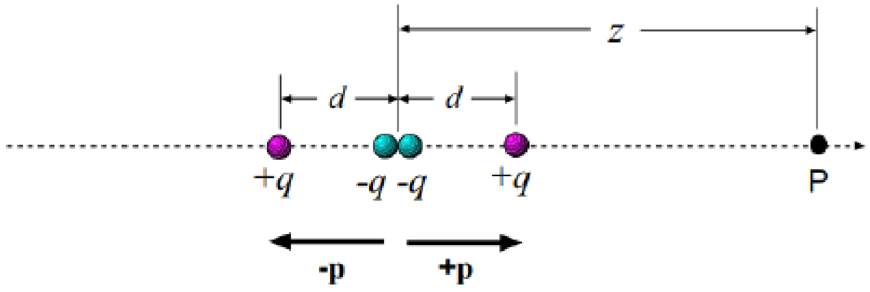

FIG. 4. Re: Q 4; The charge configuration of an electric quadrupole.

Solution. The forces acting on the mass are (i) gravity downwards, (ii) electrostatic force upward due to E. The force of gravity points downward and has magnitude mg (m is the mass of the object) and the electrical force acting on the mass has magnitude $F_{\text{elec}} = |q|E$, where $q = 24\,\mu$C is the charge of the object and E is the magnitude of the electric field. The object "floats", so the net force is zero. This gives us:

$$|q|E = mg$$

Solving for m

$$m = \frac{|q|E}{g} = 1.5 \times 10^{-3}\text{kg} = 1.5\text{g}$$

7. **Calculation of B for given current distributions.**

(a) A long, rigid wire lying along the y-axis carries a 5.0 A current flowing in the positive y direction. (a) If a constant magnetic field of magnitude 0.30 T is directed along the positive x-axis, what is the magnetic force per unit length on the wire? (b) If a constant magnetic field of 0.30 T is directed 30 degrees from the $+x$-axis towards the $+y$-axis, what is the magnetic force per unit length on the wire?

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Solution. The magnetic force on a current-carrying wire in a magnetic field is given by $\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}$. For part a, since the current and magnetic field are perpendicular in this problem, we can simplify the formula to give us the magnitude and find the direction through the Right-hand-rule. The angle θ is 90 degrees, which means $\sin\theta = 1$. Also, the length can be divided over to the left-hand side to find the force per unit length. Using this, we get:

$$F = ILB\sin\theta$$

$$\implies F/l = 1.5 \text{ N/m}$$

pointing along $\hat{k}$.

For part (b), the current times length is written in unit vector notation, as well as the magnetic field. After the cross product is taken, the directionality is evident by the resulting unit vector:

$$\vec{F} = I\vec{L} \times \vec{B}$$

$$\implies F/l = 1.30\hat{k} \text{ N/m}$$

(b) A circular current loop of radius R carrying a current I is placed in the xy-plane. A constant uniform magnetic field cuts through the loop parallel to the y-axis. Find the magnetic force on the upper half of the loop, the lower half of the loop, and the total force on the loop.

Solution. The force due to an infinitesimal line element $\vec{dl}$ on the ring is given by:

$$\begin{aligned} d\vec{F} &= IBdl\sin\theta \\ &= IB(Rd\theta)\sin\theta\,, \end{aligned}$$

where θ is the angle between $\vec{dl}$ and $\vec{B} = B\hat{y}$ and ranges from 0 to

$$\pi$$

for the upper half, π to 0 for the lower half, and 0 to 2π for the full range. Carrying out the integral leads to:

$$\begin{aligned} F^{\text{upper}} &= 2IBR \\ F^{\text{lower}} &= -2IBR \\ F^{\text{total}} &= 0 \end{aligned}$$

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(c) What is magnetic field produced by an infinite wire carrying current I at a distance d from the wire.

Solution. The magnitude of the B field is given by:

$$B = \frac{\mu_0 I}{2d}\,.$$

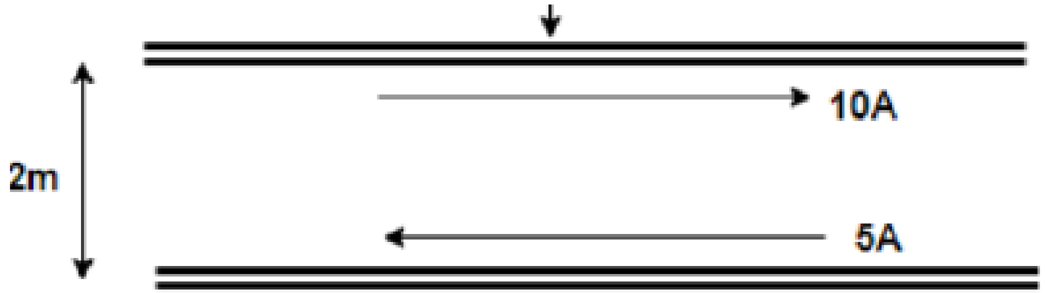

FIG. 5. For Q8(b)

The direction of the field is such that when we curl our right hand (finder) along the field, the thumb points along the current.

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(d) Use the above to calculate the magnitude of the magnetic field produced by a straight current-carrying conductor at a distance of 2 m from it when the current through the conductor is 10A. What about the direction of B.

Solution. $B = 10^{-6}$ A/m.

(e) A straight current-carrying conductor is carrying a current of 10 A and another conductor parallel to it carries a current of 5 A on the opposite side as shown in the figure below. Find the magnitude of the magnetic field produced by the system at a distance of 2 m.

Solution. Use superposition to get the magnitude $B = 7.5 \times 10^{-7}$ A/m.

(f) Show that a charge particle moving perpendicular to an external magnetic field undergoes a circular motion with angular frequency $\omega = qB/m$, where q, B and m are the charge on the particle, the strength of the magnetic field, and mass of the particle.

8. Explicitly derive the boundary conditions for $\mathbf{E}$ and $\mathbf{B}$ at an interface between two media.

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