Module :- 4

Q1) a) Show that $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$.

Ans: $\lim_{z\to 0} \frac{(x-iy)^2}{(x+iy)} = \lim_{z\to 0} \frac{x^2-y^2-2xyi}{x+iy}$.

let $y = mx$.

$\lim_{z\to 0} \frac{x^2-m^2x^2-2x^2mi}{x+imx} = \lim_{z\to 0} \frac{x^{\cancel{2}}(1-m^2-2mi)}{\cancel{x}(1+mi)}$

Now $z = x+iy$.

As $z\to 0 \Rightarrow x+iy \to 0 \Rightarrow x+imx \to 0$.

$\therefore \lim_{x(1+im)\to 0} \frac{x(1-m^2-2mi)}{(1+mi)} = 0 \times \text{finite} = 0$.

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b) $\lim_{z\to z_0} \bar{z} = \bar{z}_0$

Ans:- let $z = x+iy$.

$z_0 = x_0 + iy_0$.

$\bar{z} = x - iy$.

$\lim_{z\to z_0} \bar{z} = x_0 - iy_0 = \bar{z}_0$.

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c) let a, b and c denote complex constants. Show that $\lim_{z\to 1-i} [x+i(2x+y)] = i+1$.

Ans: $z = x+iy$.

$z \to 1-i$

Comparing above 2 equations.

$x\to 1$; $y \to -1$.

$$\lim_{z \to 1-i} [x + i(2x+y)] = \lim_{z \to 1-i} [1 + i(2-1)] = 1+i.$$

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d) Show that the limit of the func $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ as z tends to 0 does not exist.

Ans: $f(z) = \left(\frac{z}{\bar{z}}\right)^2$

$$\lim_{z \to 0} \left(\frac{x+iy}{x-iy}\right)^2$$

let $y = mx$.

$$\lim_{x(1+im) \to 0} \left[\frac{1+im}{1-im}\right]^2$$

limit depends on m.

$\therefore$ limit doesn't exist.

Q2a) Prove that $f(z) = z^2$ is continuous at z_0.

Ans: $f(z) = z^2$.

$f'(z) = 2z$.

As $z \to z_0$.

$f'(z) = 2z_0$

$\therefore$ $f'(z)$ exists as $z \to z_0$

$\therefore$ $f(z)$ is continuous at z_0.

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b) Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$.

Ans:- $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$

g $\lim_{z \to i} \dfrac{f(z) - f'(z)}{z - i}$

$= \lim_{z \to i} \dfrac{\left[\left(\dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}\right) - \left(12z^3 - 6z^2 + 16z - 2\right)\right]}{z - i}$

$= \lim_{z \to i} \dfrac{\left(\dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{\cancel{z - i}}\right) - (4 + 4i)(z - i)}{(z - i)^2}$.

$= \lim_{z \to i} \dfrac{12z^3 - 6z^2 + 16z - 2 \quad - (4 + 4i)}{2(z - i)^2}$

$= \lim_{z \to i} \dfrac{36z^2 - 12z + 16 \quad - \cancel{(4 + 4i)}}{2}$

$= -10 - 6i$

Hence limit exists at $z = i$.

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Q3a) Show that $f(z) = \bar{z}$ is non-analytic anywhere.

Ans:- $f(z) = \bar{z}$; $z = x + iy$.

$\lim_{h \to 0} f'(z) = \dfrac{f(z + h) - f(z)}{h}$.

$= \dfrac{\bar{z}\bar{h} - \bar{z}}{h}$ $\qquad h = h_1 + i h_2$

$$\lim_{h_1+ih_2 \to 0} f'(z) = \frac{h_1 - ih_2}{h_1 + ih_2}.$$

∴, limit doesn't exist, i.e. not differentiable.

——— × ——— × ———

b) Show that continuity does not imply differentiability by considering the func $f(z) = |z|^2$.

Ans:- $f(z) = |z|^2.$

$= z\bar{z}.$

$$\lim_{h \to 0} f'(z) = \frac{f(z+h) - f(z)}{h}$$

$$= \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \frac{z\bar{h}}{h} + \bar{z} + \bar{h}$$

$$= \frac{z\bar{h}}{h} + \bar{z}$$ → analytic at 0 only.

Now, $\lim_{z \to z_0} f(z) = x^2 + y^2$ let $y = mx$.

$$\lim_{x \to 0} x^2(1+m^2) = 0.$$

Hence prooved.

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c) Using the definition find the derivative of $w = f(z) = z^3 - 2z$

at the point where i) $z = z_0$ and ii) $z = -1$.

Ans:-

i) $$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

$$= \lim_{z \to z_0} \frac{z^3 - 2z - z_0^3 + 2z_0}{z - z_0}.$$

$$\lim_{h \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\lim_{h \to 0} \frac{z^3 + (z_0 + h)^3 + 3z^2(z_0 + h) + 3z(z_0 + h)^2 - 2z_0 + h - z^3}{z_0 + h}.$$

Similarly on solving we get $3z_0^2 - 2$.

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ii) $z = -1$

$f(z) = z^3 - 2z$

$z = -1 + h.$

$$\frac{z^3 - 2z}{z + 1} = 1$$

$$\lim_{z \to -1} \frac{z^3 - 2z - 1}{z + 1} = \lim_{z \to -1} 3z^2 - 2 = 1.$$

Q4 a) Show that each of these Functions is nowhere analytic

a) $f(z) = xy + iy$.

$u_x = y$.
$u_y = x$.
$v_x = 0$
$v_y = 1$

∴ not analytic as Cauchy Rieman eq not satisfied.

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b) $f(z) = 2xy + i(x^2 - y^2)$

$u_x = 2y$.
$u_y = 2x$
$v_x = 2x$
$v_y = -2y$.

Cauchy Riemann eq not satisfied.
∴ not analytic

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c) $f(z) = e^y \cdot e^{ix}$.

$= e^y [\cos x + i \sin x]$.

$= e^y \cos x + i e^y \sin x$.

$u_x = e^y(-\sin x)$
$u_y = e^y \cos x$

$v_x = e^y \cos x$
$v_y = e^y \sin x$.

not analytic

Q5) verify whether the func

$$f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z\neq 0 \\ 0 & z=0 \end{cases}$$ is non-analytic at $z=0$.

Ans:-

$$f(z) = \frac{x^3y^2}{x^6+y^2} - \frac{ix^4y}{x^6+y^2}$$

$$u_x = \frac{(x^6+y^2)(3x^2y^2) - x^3y^2 \cdot 6x^5}{(x^6+y^2)^2}.$$

$$v_x = \frac{(x^6+y^2)(4x^3y) - x^4y \cdot 6x^5}{(x^6+y^2)^2}.$$

$$v_y = \frac{(x^6+y^2)(-x^4) - x^4y(2y)}{(x^6+y^2)^2}.$$

As $v_y \neq u_x$.

$\therefore$ Func is non analytic

Q6) let u and v denote the real and imaginary components of the func f defined by means of the eq

$f(z) = \dfrac{\bar{z}^2}{z}$ when $z\neq 0$ and $f(z)=0$ when $z=0$.

Ans:- $f(z) = \dfrac{(\bar{z})^2}{z} = \dfrac{(x-iy)^2}{(x+iy)} = \dfrac{x^2-y^2-2xyi}{x+iy} \times \dfrac{x-iy}{x-iy}.$

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$$f(z) = \frac{x^3 - xy^2 - 2x^2yi - x^2yi + iy^3 - 2xy^2}{x^2 - y^2}.$$

$$f(z) = \underbrace{\frac{x^3 - 3xy^2}{x^2 - y^2}}_{(u)} + i\underbrace{\left(\frac{y^3 - 3x^2y}{x^2 - y^2}\right)}_{(v)}$$

$$u_x = \frac{(x^2-y^2)(3x^2-3y^2) - (x^3-3xy^2)\cdot 2x}{(x^2-y^2)^2}.$$

$$= 3.$$

$$u_y = \frac{(x^2-y^2)(-6xy) - (x^3-3xy^2)(-2y)}{(x^2-y^2)^2}.$$

$$= 0.$$

$$v_x = \frac{(x^2-y^2)(-6xy) - (y^3-3x^2y)(2x)}{(x^2-y^2)^2}.$$

$$= 0.$$

$$v_y = \frac{(x^2-y^2)(3y^2-3x^2) - (y^3-3x^2y)(-2y)}{(x^2-y^2)^2}.$$

$$= -3.$$

$\therefore$ Not satisfied.

Q7) Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ when

i) $u(x,y) = 2x(1-y)$

$$\frac{\delta u}{\delta x} = 2(1-y) \qquad \frac{\delta u}{\delta y} = -2x.$$

$$\frac{\delta^2 u}{\delta x^2} = 0. \qquad \frac{\delta^2 u}{\delta y^2} = 0.$$

Now, $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0. \rightarrow$ Harmonic.

Now according to Cauchy Rieman eq :-

$$u_x = v_y.$$
$$u_y = -v_x.$$

$$\frac{dv}{dy} = 2 - 2y. \qquad \frac{dv}{dx} = 2x.$$

$$v = 2y - y^2 + \phi(x)$$

$$\frac{dv}{dx} = \phi'(x) = 2x$$
$$\phi(x) = x^2 + c.$$

Now, $v(x,y) = 2y - y^2 + x^2 + c.$

~~Q9) Proof that $\frac{\delta^2 u}{\delta r^2} + \frac{1}{r}\frac{\delta u}{\delta r} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} = 0.$~~

~~and $\frac{\delta^2 v}{\delta r^2} + \frac{1}{r}\frac{\delta v}{\delta r} + \frac{1}{r^2}\frac{\delta^2 v}{\delta \theta^2} = 0.$~~

ii) $u(x,y) = 2x - x^3 + 3xy^2.$

Ans:- $\frac{\delta u}{\delta x} = 2 - 3x^2 + 3y^2.$

$$\frac{\delta^2 u}{\delta x^2} = -6x$$

$$\frac{\delta u}{\delta y} = 6xy. \qquad \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0.$$

$$\frac{\delta^2 u}{\delta y^2} = 6x$$

Now according to cauchy Riemann eq

$u_x = v_y$

$u_y = -v_x$.

$\frac{dv}{dy} = 2 - 3x^2 + 3y^2$ | $v = 2y - 3x^2y + y^3 + \phi(x)$

$\frac{dv}{dx} = -6xy$ | $\frac{dv}{dx} = -6xy + \phi'(x)$

$\phi(x) = C$.

$v = 2y - 3x^2y + y^3 + c$

—— x —— x ——

Q8) Proof that $\frac{\delta^2 u}{\delta r^2} + \frac{1}{r}\frac{\delta u}{\delta r} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} = 0$. and

$\frac{\delta^2 v}{\delta r^2} + \frac{1}{r}\frac{\delta v}{\delta r} + \frac{1}{r^2}\frac{\delta^2 v}{\delta \theta^2} = 0$

Ans:- $Z = x + iy = re^{i\theta}$.

$u + iv = f(z) = f(re^{i\theta})$

$\Rightarrow \frac{\delta u}{\delta r} + i\frac{\delta v}{\delta r} = f'(re^{i\theta}) \cdot e^{i\theta}$.

$\frac{\delta u}{\delta \theta} + \frac{i\delta v}{\delta \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir\left[\frac{\delta u}{\delta r} + \frac{i\delta v}{\delta r}\right]$.

Equating imaginary and real part

$\frac{\delta u}{\delta r} = \frac{1}{r}\frac{\delta v}{\delta \theta}$

$\frac{\delta v}{\delta r} = -\frac{1}{r}\frac{\delta u}{\delta \theta}$

$\frac{\delta^2 u}{\delta r^2} = -\frac{1}{r^2}\frac{\delta v}{\delta \theta} + \frac{1}{r}\frac{\delta^2 v}{\delta \theta \delta r}$ | $\frac{\delta^2 u}{\delta \theta^2} = -r\frac{\delta^2 v}{\delta r \delta \theta}$

$$\frac{\delta^2 u}{\delta r^2} + \frac{1}{r}\frac{\delta u}{\delta r} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} = 0.$$

Now:- $\frac{\delta^2 v}{\delta r^2} = \frac{1}{r^2}\frac{\delta u}{\delta \theta} - \frac{1}{r}\frac{\delta^2 u}{\delta r \delta \theta}.$

Similarly:- $\frac{\delta^2 v}{\delta r^2} + \frac{1}{r}\frac{\delta v}{\delta r} + \frac{1}{r^2}\frac{\delta^2 v}{\delta \theta^2} = 0.$

x —— x

Q9) If a func $f(z) = u(x,y) + iv(x,y)$ is analytic in a Dom D then its component functions u and v are harmonic in

Ans:- $f(z) = u(x,y) + iv(x,y)$

Given z is analytic

ie $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ | $\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 v}{\delta x \delta y}$

$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ | $\frac{\delta^2 u}{\delta y^2} = -\frac{\delta^2 v}{\delta x \delta y}.$

$u_{xx} + u_{yy} = 0.$

$$\left.\begin{aligned}\frac{\delta^2 v}{\delta y^2} &= \frac{\delta^2 u}{\delta x \delta y}\\ \frac{\delta^2 v}{\delta x^2} &= -\frac{\delta^2 u}{\delta x^2}\end{aligned}\right\} \Rightarrow \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0.$$

x —— x

Q10) For the Functions f and Contours C, use parametric representations for C to evaluate $\int_C f(z)\,dz$ where $f(z) = (z+2)/z$ and C is

a) Semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$

$$\int f(z)\,dz = i\int_0^{\pi} \frac{e^{i\theta}+2}{e^{i\theta}} \times e^{i\theta}\, d\theta.$$

$$= i\int_0^{\pi} (e^{i\theta}+2)\, d\theta.$$

$$= i\left[e^{i\pi} + 2\pi\right].$$

b) The circle, $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$

$$\int f(z)\,dz = i\int \frac{2e^{i\theta}+2}{2e^{i\theta}} \cdot 2e^{i\theta}\, d\theta.$$

$$= 2ie^{i\pi} + 4i\pi.$$

——— x ——— x ———

Q11) Apply Cauchy's theorem to show that $\int_C f(z)\,dz = 0$ when the Contour C is unit circle $|z|=1$ in either direction and when

a) $f(z) = \dfrac{z^3}{z-3}$

As $f(z)$ is analytic at $|z|=1$ so by Cauchy's theorem

$$\int_C f(z)\,dz = 0.$$

——— x ——— x ———

b) $f(z) = ze^{-z}$.

As $f(z)$ is analytic at $|z|=1$ So by cauchy's theorem

$\int_C f(z)\,dz = 0.$

——— x ——— x ———

c) $f(z) = \frac{1}{z^2+z+2}$

At $|z|=1$, $f(z)$ is analytic so by applying cauchy's theorem we get

$$\oint_c f(z)\, dz = 0.$$

x ——— x

Q12) Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in positive sense when

a) $g(z) = \frac{1}{z^2+1}$.

Singularities are at $z_0 = \pm i$

$$\int g(z)\, dz = \int \frac{1}{(z+i)(z-i)}\, dz.$$

$$= \int \frac{1}{2i}\left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right] dz.$$

$$= \frac{1}{2i}\left[\int \frac{dz}{(z-i)} - \int \frac{dz}{(z+i)}\right.$$

$$= \frac{1}{2i}\left[2\pi i\, f(i) - 2\pi i\, f(-i)\right].$$

$$= 0.$$

x ——— x

b) $g(z) = \frac{1}{(z^2+4)^2}$. Singularities are at $z_0 = \pm 2i$. Only $+2i$ lies in $|z-i|=2$.

$$\int g(z)\, dz = \int \frac{1}{(z^2+4)^{1+1}}\, dz = \frac{2\pi i}{1!} \frac{d}{dz}(z^2+4)\Big|_{z=2i} = -8\pi$$

Q13) Write down the laurent series in powers of z that represent the ~~Abve~~ func $f(z) = \frac{1}{z(1+z^2)}$ in the domain $0<|z|<1$.

$f(z) = \frac{1}{z(1+z^2)}$ $|z|<1$.

$= \frac{1}{z(1+z^2)} = \frac{1}{z}(1-z^2+z^4-z^6+\dots)$

$= \left[\frac{1}{z} - z + z^3 - z^5 + \dots\right]$

Q14) Find the residue at $z=0$ of the func :-

a) $f(z) = \frac{1}{z+z^2}$.

$f(z) = \frac{1}{z(1+z)}$

0 is a pole of order 1

$\text{Res}[f(z), z_0] = \lim_{z \to z_0} (z-z_0) f(z)$

$= \lim_{z \to 0} z \times \frac{1}{z(z+1)}$

$= 1.$

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b) $f(z) = \frac{z - \sin z}{z}$.

~~removable singu~~

~~0 is a pole of order 1.~~

0 is a removable singularity.

Hence $\text{Res}[f(z), z_0] = 0$.

Q15) Use residue theorem to evaluate the integrals of each of these func around $|z| = 3$.

a) $f(z) = \frac{e^{-z}}{z^2}$.

0 is a pole of order 2.

$\int f(z)\,dz = 2\pi i \, Res[f(z), z_0]$.

$= 2\pi i \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz} \frac{z^2 \cdot e^{-z}}{z^2}$

$= 2\pi i \cdot \frac{1}{1} \lim_{z \to 0} -e^{-z}$.

$= -2\pi i$.

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b) $f(z) = \frac{(z+1)}{z(z-2)}$

0 and 2 are poles of order 1. in $|z| = 3$.

$\int f(z)\,dz = 2\pi i \left[Res(f(z), 0) + Res(f(z), 2) \right]$.

$= 2\pi i \left[\lim_{z \to 0} \frac{\cancel{z}(z+1)}{\cancel{z}(z-2)} + \lim_{z \to 2} \frac{\cancel{(z-2)}(z+1)}{z\cancel{(z-2)}} \right]$.

$= 2\pi i \left[-\frac{1}{2} + \frac{3}{2} \right] = 2\pi i$.

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c) $f(z) = z^2 e^{1/z}$.

As the func is analytic in $|z| = 3$ so by applying Cauchy's theorem

$\int f(z)\,dz = 0$.