Power Series..

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n =$$

$$P_0(x) \frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) \cdot y = 0.$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)} \frac{dy}{dx} + \frac{P_2(x)}{P_0(x)} \cdot y = 0.$$

Behaviour of eq^n (1) near point $x = x_0$ depends upon behaviour of coefficient functions $P(x)$ & $Q(x)$. If $P(x)$ & $Q(x)$ are analytic at x_0, i.e. each has a power series expansion valid in some neighbourhood of this point x_0 is called ordinary point of (1).

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + q \cdot y = 0. \quad ; \quad P = \frac{P_1(x)}{P_0(x)}$$

$q =$

*If $P(x)$ or $Q(x)$ are not analytic at $x = x_0$, i.e. $P(x)$ or $Q(x)$ is infinity, then ~~they~~ x_0 is singular point.

* If $P(x)$ &

$Q_1(x) = (x-x_0) P(x)$ &

$Q_2(x) = (x-x_0)^2 q(x)$

* If Q_1 & Q_2 are analytic (not ∞), at $x = x_0$, 'x_0' is called regular singular point.

$$P_0(x) \cdot \frac{d^2y}{dx^2} + P_1(x) \cdot \frac{dy}{dx} + P_2(x) \cdot y = 0.$$

ordinary point $P_0(0) \neq 0$

singular point $P_0(0) = 0$.

Ex: $\frac{d^2y}{dx^2} + y = 0$.

$P_0(x) = 1$

$P_0(0) \neq 0$

$\therefore x = 0$, is an ordinary point.

Q $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$.

$P_0(x) = 9x(1-x)$

$P_0(0) = 0$; at $x_0 = 0$

$\therefore x_0 = 0$ is a singular point.

Q $x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$.

$P_0(x) = x$

At $x = 0$,

$P_0(0) = 0$

$\therefore$ singular point

Find regular singular point of the differential eqn:

$$x^2(x-2)^2 + 2(x-2)\frac{dy}{dx} + (x+3)y = 0$$

$$P(x) = \frac{P_1(x)}{P_0(x)} = \frac{2\cancel{(x-2)}}{x^2(x-2)^{\cancel{2}}} = \frac{2}{x^2(x-2)}$$

$$q(x) = \frac{P_2(x)}{P_0(x)} = \frac{(x+3)}{x^2(x-2)^2}$$

$P(x)$ & $q(x)$ are not analytic at $x=0$ & $x=2$, so singular points.

$x_0 = 0, 2$

$Q_1(x) = (x-0)\cdot P(x)$

$$= \frac{\cancel{x}\cdot 2}{x^{\cancel{2}}(x-2)}$$

$$Q_1(x) = \frac{2}{x(x-2)}$$

At $x=0$, $Q_1(0) = \infty$

$Q_2(x) = (x-0)^2 \cdot q(x)$

$$Q_2(x) = \frac{x^2(x+3)}{(x-2)^2}$$

At $x=0$,

$Q_2(x) = 0 \neq \infty$

$x=0$.

$$Q_1(x) = \frac{\cancel{(x-2)}\cdot 2}{x^2\cancel{(x-2)}} = \frac{1}{2} \neq \infty$$

$$Q_2(x) = \frac{\cancel{(x-2)^2}\cdot(x+3)}{x^2\cancel{(x-2)^2}} = \frac{5}{4} \neq \infty$$

$x_0 = 2$

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* $P_0(0) \neq 0$: ordinary point

* $P_0(0) = 0$: singular point

* $P(x) = \dfrac{P_1(x)}{P_0(x)}$ & $q(x) = \dfrac{P_2(x)}{P_0(x)}$

If $P(x)$ ~~&~~ or $q(x)$ come out to be ∞, the function is non-analytic or is singular.

If, $P(x) = \infty$ or $q(x) = \infty$: Non-analytic / singular.

$$\left.\begin{array}{l} * \ Q_1(x) = p(x) \cdot (x - x_0) \neq \infty \\ \quad \text{or} \\ Q_2(x) = \cancel{P(x)}\, q(x)\,(x - x_0) \neq \infty \end{array}\right\}$$ analytic function or regular singular point.

Series solution:

Ordinary point ($x=0$)

Assume series solution to be
$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$= a_0 + a_1 x + a_2 x^2 \ldots$$

Regular singular point

Assume series solution to be
$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$
$$= x^m (a_0 + a_1 x + a_2 x^2 + \ldots)$$

Series solution when $x=0$ is an ordinary point

Let the series be:

$$P_0(x)\, y'' + P_1(x)\, y' + P_2(x)\, y = 0 \quad \text{—— (1)}$$

P_0, P_1 & P_2 are polynomials in 'x'.

& $P_0(0) \neq 0$ ($\because$ 0 is an ordinary point)

Step 1:

Let the solution be of the form: — ②

$y = a_0 + a_1 x + a_2 x^2 + \cdots$

Step 2:

Finding $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ from ②,

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$$

Replace these values in ②.

Step 3: Equate to '0' the coefficients of various powers of 'x' & determine $a_2, a_3, a_4, \cdots$ in terms of a_0 & a_1

The result obtained by equating to 0, the coefficient of x^n is called recurrence relation.

Step 4: Substituting the values of $a_2, a_3, a_4 \cdots$ in ②, we get the desired series solution, having a_0 & a_1 as ~~series solution have~~ arbitrary constants.

Solve the following equation in series: 2

Q.1 $\frac{d^2y}{dx^2} + y = 0$ — ④

$P_0(x) = 1$

$\therefore P_0(0) = 1 \neq 0$

$\therefore$ '0' is an ordinary point.

$\therefore$ soln: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ — ①

$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n \cdot a_n x^{n-1}$ — ②

$+ (n+1)\, a_{n+1} \cdot x^n$

$+ (n+2)\, a_{n+2} \cdot x^{n+1}$

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$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + \cdots + n(n-1)x^{n-2} \quad (3)$$

$$12a_4x^2 + \cdots \qquad + n(n+1)\cdot a_{n+1}\cdot x^{n-1} + (n+2)(n+1)a_{n+2}\cdot x^n$$

Replacing

$$= (2a_2 + 6a_3x + \cdots n(n-1)x^{n-2}) + (a_0 + a_1x + a_2x^2 + a_3x^3 \cdots + a_nx^n)$$

$$= (2a_2 + a_0) + x(6a_3 + a_1) + x^2(12a_4 + a_2) + x^3(\;-\;) + \cdots$$

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -a_0/2$$

$\therefore 6a_3 + a_1 = 0$; $12a_4 + a_2 = 0$

$$a_3 = \frac{-a_1}{6} \qquad a_4 = \frac{-a_2}{12} = \frac{a_0}{24}.$$

$$\therefore y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$= a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2\cdot3}\cdot x^3 + \frac{a_0}{2\cdot3\cdot4}\cdot x^4 \cdots$$

$$= \left(a_0 - \frac{a_0}{2}x^2 + \frac{a_2}{2\cdot3\cdot4}x^4 \cdots\right) + a_1\left(x - \frac{x^3}{2\cdot3} + \frac{x^5}{2\cdot3\cdot4\cdot5} \cdots\right)$$

2. $\frac{d^2y}{dx^2} + xy = 0$

$P_0(x) = 1 \neq 0$, $\therefore$ ordinary point

$\therefore$ solution of the eqn :

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2}$$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + 7a_7x^6 + \cdots n\cdot a_n\cdot x^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1}$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42\cdot a_7x^5 + \cdots n(n+1)a_{n+1}x^{n-1} + (n+1)(n+2)a_{n+2}x^n$$

$$y''(2a_2+6a_3x+12a_4x^2+\cdots) + x(a_0+a_1x+a_2x^2+\cdots + a_nx^n) = 0$$

$$= (2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+42a_7x^5+\cdots) + x(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5\cdots)$$

$$= 2a_2 + x(6a_3+a_0) + x^2(12a_4+a_1) + x^3(20a_5+a_2) + x^4(30a_6+a_3) + x^5(42a_7+a_4) + \cdots + x^n((n+1)(n+2)a_{n+2}+a_{n-1})$$

$\therefore 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

$6a_3 + a_0 = 0 \Rightarrow \boxed{a_3 = \frac{-a_0}{6}}$

$12a_4 + a_1 = 0 \Rightarrow \boxed{a_4 = \frac{-a_1}{12}}$

$20a_5 + a_2 = 0 \Rightarrow \boxed{a_5 = 0}$

$30a_6 + \frac{-a_0}{6} = 0 \Rightarrow a_6 = \frac{a_0}{6\times30} \Rightarrow \boxed{a_6 = \frac{a_0}{180}}$

$\therefore y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

$= a_0 + a_1x - \frac{a_0}{6}\cdot x^3 - \frac{a_1}{12}\cdot x^4 + \frac{a_0}{180}\cdot x^6 \cdots$

$= a_0\left(1 - \frac{x^3}{6} + \frac{x^6}{180} \cdots\right) + a_1\left(x - \frac{x^4}{12} + \cdots\right)$

$f \quad (n+1)(n+2)a_{n+2} + a_{n-1} = 0$

$$\boxed{\therefore a_{n+2} = \frac{-a_{n-1}}{(n+1)(n+2)}}$$

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3. $\frac{d^2y}{dx^2} + x^2 y = 0$.

$P_0(x) = 1$

$\because P_0(0) \neq 0$, $\therefore$ ordinary point.

Thus, $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \dots a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2}$.

$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6 + 8a_8 x^7 \dots n \cdot a_n x^{n-1} + (n+1) a_{n+1} x^n + (n+2) a_{n+2} x^{n+1}$

$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \dots$

$56a_8 x^6 + \dots = n(n-1) a_n x^{n-2} + n(n+1) a_{n+1} x^{n-1} +$

$(n+1)(n+2) a_{n+2} \cdot x^n$.

$\therefore (2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$
$30a_6 x^4 + 42a_7 x^5 + \dots) + x^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots a_n x^n) = 0$

$2a_2 + x(6a_3) + x^2(12a_4 + a_0) + x^3(20a_5 + a_1) + x^4(30a_6 + a_2)$
$+ x^5(42a_7 + a_3) + x^6(56a_8 + a_4) + x^7(72a_9 + a_5) \dots = 0$
$+ x^8(90a_{10} + a_6) + x^9(110a_{11} + a_7) + \dots x^n((n+1)(n+2)a_{n+2} + a_{n-2})$

$\therefore 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$ (coefficient of $x^0 = 0$)

$6a_3 = 0 \Rightarrow \boxed{a_3 = 0}$ (coefficient of $x^1 = 0$)

$12a_4 + a_0 = 0 \Rightarrow \boxed{a_4 = \frac{-a_0}{12}}$ (coefficient of $x^2 = 0$)

$20a_5 + a_1 = 0 \Rightarrow \boxed{a_5 = \frac{-a_1}{20}}$

$30a_6 + a_2 = 0 \Rightarrow a_6 = \frac{-a_2}{30} = 0$ $(\because a_2 = 0)$

$42a_7 + a_3 = 0$

$\Rightarrow a_7 = \frac{-a_3}{42} = 0 \; (\because a_3 = 0)$

$90 a_{10} + a_6 = 0$

$\boxed{a_{10} = \frac{-a_6}{90} = 0}$

$110 a_{11} + a_7 = 0$

$\boxed{a_{11} = 0}$

$56 a_8 + a_4 = 0$

$a_8 = \frac{-a_4}{56} = \frac{-1}{56}\left(\frac{-a_0}{12}\right) = \frac{a_0}{56 \times 12}$

$12 \times 11 \, a_{12} + a_8 = 0$

$a_{12} = \frac{-a_8}{12 \times 11}$

$72 a_9 + a_5 = 0$

$a_9 = \frac{-a_5}{72}$

$= \frac{-1}{12 \times 11}\left(\frac{a_0}{56 \times 12}\right)$

$a_9 = \frac{-1}{72}\left(\frac{-a_1}{20}\right) = \frac{a_1}{72 \times 20}$

$\boxed{a_{12} = \frac{-a_0}{12 \times 12 \times 11 \times 56}}$

$\therefore y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \ldots$

$y = a_0 + a_1 x + 0 \cdot x^2 + 0 \cdot x^3 - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + 0 \cdot x^6 + 0 \cdot x^7 + \frac{a_0}{56 \times 12} x^8 + \frac{a_1 x^9}{72 \times 20} + \ldots$

$y = a_0\left(1 - \frac{x^4}{12} + \frac{x^8}{56 \times 12} - \ldots\right) + a_1\left(x - \frac{x^5}{20} + \frac{x^9}{72 \times 20} - \ldots\right)$

$a_0\left(1 - \frac{x^4}{12} + \frac{x^8}{12 \times 56} - \frac{x^{12}}{12 \times 56 \times 11 \times 12} + \ldots\right) + a_1\left(x - \frac{x^5}{20} + \frac{x^9}{20 \times 72} - \frac{x^{13}}{12 \times 13 \times 72 \times 20} + \ldots\right)$

$y = a_0\left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot}\right.$

Series solution when $x=0$ is a regular singular point:

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad \text{--- (1)}$$

→ If $x=0$, is a regular singular point then solution is assumed to be:

$$y = x^m(a_0 + a_1 x + a_2 x^2 - - a_n x^n)$$

$$= \sum_{n=0}^{\infty} a_n x^{m+n}.$$

→ find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ & substitute in eq^n (1).

→ Equate the coefficients of powers of x to 0.

→ Express equation in terms of 'a_0'. ($a_0 \neq 0$)

→ complete solution depends on the nature of the roots of the indicial equation.

Case 1: Roots of the indicial equations are distinct & do not differ by ~~an~~ integer:

$$y = c_1 y_1 \Big|_{m_1} + c_2 y_2 \Big|_{m_2}$$

Case 2: Roots of the indicial equation are ~~zero~~ equal

$$y = c_1 (y_1)_{m_1} + c_2 \left(\frac{\partial y_1}{\partial m}\right)_{m_1}$$

Case 3: Roots of the indicial equations are distinct & differ by an integer making coefficients of 'y' infinite.

let m_1 & m_2 be the roots such that $m_1 < m_2$. If some coefficient of 'y' series becomes infinite when $m = m_1$, modify the form of 'y' by replacing $a_0 = b_0(m - m_1)$

$\therefore y = c_1 (y_1)_{m_2} + c_2 \left(\frac{\partial y_1}{\partial m}\right)_{m_1}$

Q. solve in series the equation

$$9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$$

$P_0(x) = 9x(1-x)$

$\therefore P_0(0) = 0$

$\therefore$ singular point.

$Q_1(x) = (x - x_0)\,P(x)$

$= (x-0)\dfrac{(-12)^4}{{}_3 9x(1-x)}$

$= \dfrac{-4x}{3x(1-x)} = \dfrac{-4}{3(1-x)}$

$Q_2(x) = (x - x_0)^2 \cdot Q(x)$

$= (x-0)^2 \cdot \dfrac{4}{9x(1-x)}$

$= \dfrac{4x^2}{9x(1-x)}$

$= \dfrac{4x}{9(1-x)}$.

$\therefore Q_1(0) = -4/3$ & $Q_2(0) = 0$

$\neq \infty$ $\qquad$ $\neq \infty$.

$\therefore x=0$ is Analytic or regular singular.

$\therefore$ solution is:

$y = x^m(a_0 + a_1x + a_2x^2 \cdots + a_nx^n)$

$\dfrac{dy}{dx} = m\cdot x^{m-1}a_0 + a_1(m+1)x^m + a_2(m+2)x^{m+1} \cdots$

$\cdots a_n(m+n)x^{m+n-1}$

$\dfrac{d^2y}{dx^2} = m(m-1)x^{m-2}a_0 + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$

$+ \cdots a_n(m+n)(m+n-1)x^{m+n-2}$

$= 9x(1-x)\cdot\left[m(m-1)x^{m-2}a_0 + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m\right.$
$\left. \cdots\right]$

$-12\left[a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} \cdots\right]$

$+4\left(a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} \cdots\right]$

$= (9x - 9x^2)(a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$
$+\cdots) - 12\left[a_0 m x^{m-1} + \cdots\right] + 4\left[a_0 x^m + a_1 x^{m+1}\right.$
$\left.\cdots\right]$

$= 9a_0 m(m-1)x^{m-1} + 9a_1 m(m+1)x^m + 9a_2(m+1)(m+2)x^{m+1} + \cdots$
$- 9a_0 m(m-1)x^m - 9a_1 m(m+1)x^{m+1} - 9a_2(m+1)(m+2)x^{m+2} + \cdots$
$-12(a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots)$

$+4(a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots)$

$= x^{m-1}(9a_0 m(m-1) - 12a_0 m) + x^m(9a_1 m(m+1) - 12a_1(m+1)$
$+4a_0 + \cdots$
$-9a_0 m(m-1)) + \cdots$

lowest power of x is $(m-1)$

$9a_0 m(m-1) - 12a_0 m = 0$

$\because a_0 \neq 0,$

$9m(m-1) - 12m = 0$

$3m(m-1) - 4m = 0.$

$\boxed{\therefore m = 0}$ or $3m - 3 - 4 = 0$

$\boxed{m = 7/3}$

coefficient of $x^m = 0$

$a_1\left[9m(m+1) - 12(m+1)\right] + 4a_0 - 9a_0 m(m-1) = 0$

$3a_1\left[3m(m+1) - 4(m+1)\right] + a_0\left[4 - 9m(m-1)\right] = 0.$

$3a_1\left[3m(m+1) - 4(m+1)\right] = \left[9m(m-1) - 4\right]a_0.$

$a_1 = \dfrac{9m(m-1) - 4}{3\left[3m(m+1) - 4(m+1)\right]}a_0.$

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$$\boxed{a_1 = \frac{a_0 \cdot (3m+1)\cancel{(3m-4)}}{3(m+1)\cancel{(3m-4)}} = \frac{a_0(3m+1)}{3(m+1)}}$$

$\rightarrow$ $9a_2(m+1)(m+2) - 9a_1(m+1)m - 12a_2(m+2) + 4a_1 = 0$

$$a_2[9(m+1)(m+2) - 12(m+2)] = 9a_1(m+1)m - 4a_1$$

$$a_2 = \frac{a_1[9m(m+1) - 4]}{9(m+1)(m+2) - 12(m+2)}$$

$$a_2 = \frac{a_1[9m^2+9m-4]}{3(m+2)[3(m+1) - 4]} = \frac{a_1[9m^2+12m-3m-4]}{3(m+2)(3m+3-4)}$$

$$= \frac{a_1[3m(3m+4) - (3m+4)]}{3(m+2)(3m-1)}$$

$$= \frac{a_1[3m-1][3m+4]}{3(m+2)\cdot \cancel{3(m-3)}(3m-1)}$$

$$a_2 = \frac{a_1(3m+4)}{3(m+2)}$$

$$a_2 = \frac{(3m+4)}{3(m+2)} \cdot \frac{a_0(3m+1)}{3(m+1)}$$

$$\boxed{a_2 = \frac{a_0}{9}\frac{(3m+4)(3m+1)}{(m+1)(m+2)}}$$

$\rightarrow$ $9a_3(m+2)(m+3) - 9a_2(m+1)(m+2) - 12a_3(m+3) + 4a_2 = 0$

$$a_3[9(m+2)(m+3) - 12(m+3)] = 9a_2(m+1)(m+2) - 4a_2$$

$$a_3 = \frac{a_2[9(m+1)(m+2) - 4]}{3(m+3)[3(m+2) - 4]}$$

$$= \frac{a_2[9(m^2+3m+2) - 4]}{3(m+3)(3m+6-4)} = \frac{a_2[9m^2+27m+18-4]}{3(m+3)(3m+2)}$$

$7\times 2\times 3\times 3$

21×6

$$\frac{a_2\left[9m^2+27m+14\right]}{3(m+3)(3m+2)}$$

$$\frac{a_2\left[9m^2+21m+6m+14\right]}{3(m+3)(3m+2)}$$

$$\frac{a_2\left[3m(3m+7)+2(3m+7)\right]}{3(m+3)(3m+2)}$$

$$\frac{a_2\cancel{\left[3m+2\right]}\left[3m+7\right]}{3(m+3)\cancel{(3m+2)}} = \frac{a_2(3m+7)}{3(m+3)}$$

$$= \frac{(3m+7)}{3(m+3)}\cdot\frac{a_0}{9}\frac{(3m+4)(3m+1)}{(m+1)(m+2)}$$

$$\boxed{a_3 = \frac{a_0}{27}\cdot\frac{(3m+1)(3m+4)(3m+7)}{(m+1)(m+2)(m+3)}}$$

2.

$$\therefore y = c_1 y_1\Big|_{m_1} + c_2 y_2\Big|_{m_2}$$

$$y = \cancel{a_0 x}\; a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots$$

$$y = a_0x^m + \frac{a_0(3m+1)}{3(m+1)}x^{m+1} + \frac{a_0(3m+1)(3m+4)}{3^2(m+1)(m+2)}x^{m+2} + \frac{a_0(3m+1)(3m+4)(3m+7)}{3^3(m+1)(m+2)(m+3)}x^{m+3} + \ldots$$

$$\therefore y_1\Big|_{m_1=0} = a_0\left[\frac{1}{3}\cdot\frac{x}{1} + \frac{1\cdot 4x^2}{3^2\,1\cdot 2} + \frac{1\cdot 4\cdot 7x^3}{3^3\,1\cdot 2\cdot 3} + \ldots\right]$$

$$= a_0\left[\frac{x}{3\cdot 1} + \frac{1\cdot 4x^2}{1\cdot 2\; 3^2} + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{x^3}{3^3} + \ldots\right]$$

$$y_2\Big|_{m_2=7/3} = a_0\left[x^{7/3} + \frac{1}{\cancel{3}}\frac{8\cdot\cancel{3}}{10}x^{10/3} + \frac{1}{\cancel{3^2}}\frac{8\cdot 11}{\frac{10}{\cancel{3}}\cdot\frac{13}{\cancel{3}}}x^{13/3} + \frac{1}{\cancel{3^3}}\cdot\frac{8\cdot 11\cdot 14}{\frac{10}{\cancel{3}}\cdot\frac{13}{\cancel{3}}\cdot\frac{16}{\cancel{3}}}x^{16/3} + \ldots\right]$$

$$= a_0\left[x^{7/3} + \frac{8}{10}x^{10/3} + \frac{8\cdot 11}{10\cdot 13}x^{13/3} + \frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16}\cdot x^{16/3} + \cdots\right]$$

∴ solution:

$$y = \cancel{a}\ a_0\left[\cancel{\frac{x}{1}}\ \frac{1}{1}\cdot\frac{x^1}{3} + \frac{1\cdot 4}{1\cdot 2}\cdot\frac{x^2}{3^2} + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{x^3}{3^3} + \cdots\right]$$

$$+ a_0\left[x^{7/3} + \frac{8}{10}\cdot x^{10/3} + \frac{8\cdot 11}{10\cdot 13}x^{13/3} + \frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16}\cdot x^{16/3} + \cdots\right]$$

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Q. $2xy'' + y' + xy = 0.$

$P_0(x) = 2x$

$P_0(0) = 0.$

$\therefore$ 0 is a singular point.

$Q_1(x) = (x - x_0)\, P(x)$

$= (x-0) \cdot \frac{1}{2x}$

$= \frac{1}{2} \neq \infty$

$Q_2(x) = (x - x_0)\, q(x)$

$= \frac{x^2 \cdot \cancel{x}}{2\cancel{x}} = \frac{x^2}{2} = 0$

$\because Q_1(x)$ & $Q_2(x)$ at $x=0 \neq \infty$

$\therefore$ $x=0$ is a regular singular point.

$\therefore$ series solution is assumed to be

$y = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + \cdots$

$\frac{dy}{dx} = m \cdot a_0 x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots a_3(m+3)x^{m+2} + \cdots$

~~$\frac{d^2y}{dx^2} = m^2 a_0 x^{m-1} + a_1 m(m+1) x^m$~~

$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + a_3(m+2)(m+3)x^{m+1} + \cdots$

$$2x\left[m(m-1)a_0x^{m-2}+a_1(m+1)m\,x^{m-1}+a_2(m+1)(m+2)x^m\right.$$

$$+a_3(m+2)(m+3)x^{m+1}+(m+3)(m+4)a_4x^{m+2}+(m+4)(m+5)$$

$$a_5x^{m+3}$$

$$\left.+\cdots\right]$$

$$+ma_0x^{m-1}+(m+1)a_1x^m+a_2(m+2)x^{m+1}+a_3(m+3)x^{m+2}$$

$$+(m+4)a_4x^{m+3}+a_5(m+5)x^{m+4}+a_6(m+6)x^{m+5}+\cdots$$

$$+x\left[a_0x^m+\cancel{a_0}\,a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+a_4x^{m+4}+\right.$$

$$\left.a_5x^{m+5}+a_6\,x^{m+6}+\cdots\right].$$

$$x^{m-1}\left[2m(m-1)a_0+ma_0\right]+x^m\left[2m(m+1)a_1+(m+1)a_1\right]$$

$$+x^{m+1}\left[\cancel{2a}\;2(m+1)(m+2)a_2+(m+2)a_2+a_0\right]$$

$$+x^{m+2}\left[2a_3(m+2)(m+3)+a_3(m+3)+a_1\right]+$$

$$x^{m+3}\left[2(m+3)(m+4)a_4+(m+4)a_4+a_2\right]+x^{m+4}$$

$$\left[2(m+4)(m+5)a_5+(m+5)a_5+a_3\right]+\cdots$$

$\rightarrow$ $2m(m-1)a_0+ma_0=0$

$a_0\neq 0$

$\therefore\; 2m(m-1)+m=0$

$m=0$ or

$2(m-1)+1=0$

$m-1=-\frac{1}{2}$

$m=1-\frac{1}{2}=\frac{1}{2}.$

$m=0,\frac{1}{2}$

$y=c_1y_1|_{m_1}+c_2y_2|_{m_2}.$

$$\rightarrow \quad 2m(m+1)a_1 + (m+1)a_1 = 0$$

$$a_1\left[2m(m+1) + (m+1)\right] = 0 \Rightarrow \boxed{a_1 = 0}$$

$$\rightarrow \quad 2(m+1)(m+2)a_2 + (m+2)a_2 + a_0 = 0$$

$$a_2\left[2(m+1)(m+2) + (m+2)\right] = -a_0$$

$$a_2 = \frac{-a_0}{(m+2)\left[2(m+1)+1\right]}$$

$$\boxed{a_2 = \frac{-a_0}{(2m+3)(m+2)}}$$

$$\rightarrow \quad 2a_3(m+2)(m+3) + a_3(m+3) + a_1 = 0$$

$$a_3 = \frac{-a_1}{\left[2(m+2)(m+3) + (m+3)\right]} = 0 \Rightarrow \boxed{a_3 = 0}$$

$$\rightarrow \quad 2(m+3)(m+4)a_4 + (m+4)a_4 + a_2 = 0$$

$$a_4\left[2(m+4)(m+3) + (m+4)\right] = -a_2$$

$$a_4 = \frac{-a_2}{(m+4)\left[2(m+3)+1\right]}$$

$$= \frac{-a_2}{(m+4)(2m+7)}$$

$$= \frac{-1}{(m+4)(2m+7)} \frac{(-a_0)}{(2m+3)} = \frac{a_0}{(2m+3)(2m+7)(m+4)(m+2)}$$

$$y = a_0\left\{ a_0x^m + 0 - \frac{a_0}{(m+2)(2m+3)}x^{m+2} + 0\cdot x^{m+3} + \frac{a_0}{(m+2)(m+4)(2m+3)(2m+7)}x^{m+4}\right.$$

$$y = a_0\left[x^m - \frac{x^{m+2}}{(m+2)(2m+3)} + \frac{x^{m+4}}{(m+2)(m+4)(2m+3)(2m+7)} - \cdots\right]$$

$$y_1\Big|_{m_1=0} = a_0\left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot4\cdot3\cdot7} - \frac{x^6}{2\cdot4\cdot6\cdot3\cdot7\cdot11} + \cdots\right]$$

$$y_2\Big|_{m_2=1/2} = a_0\left[x^{1/2} - \frac{x^{5/2}}{\frac{5}{2}\cdot 4} + \frac{x^{9/2}}{\frac{5}{2}\cdot\frac{9}{2}\cdot 4\cdot 8} - \frac{x^{13/2}}{\frac{5}{2}\cdot\frac{9}{2}\cdot\frac{13}{2}\cdot 4\cdot 8\cdot 12} + \cdots\right.$$

$$a_0\left[x^{1/2} - 2\cdot\frac{x^{5/2}}{5\cdot 4} + 2^2\cdot\frac{x^{9/2}}{5\cdot 9\cdot 4\cdot 8} - 2^3\frac{x^{13/2}}{5\cdot 9\cdot 13\cdot 4\cdot 8\cdot 12} + \cdots\right]$$

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$=0$ Q.2 $2x^2y'' + x(2x+1)y' - y = 0$.

$P_0(x) = 2x^2$

$P_0(0) = 0$

$\therefore$ $x=0$ is singular point.

$Q_1(x) = (x-0)\,P(x)$

$= x \cdot \dfrac{P_1(x)}{P_0(x)} = \cancel{x} \cdot \dfrac{\cancel{x}(2x+1)}{2\cancel{x^2}} = 1/2$

$Q_2(x) = \dfrac{(\cancel{x}-0)^2(-1)}{2x^2} = \dfrac{-1}{2x}$

∴ solution:

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \text{------})$$

$$\frac{dy}{dx} = a_0m\cdot x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \ldots + a_4(m+4)x^{m+3} + \ldots$$

$$\frac{d^2y}{dx^2} = a_0m(m-1)x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + a_3(m+3)(m+2)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \ldots$$

$$\therefore 2x^2\left[a_0m(m-1)x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + a_3(m+3)(m+2)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \ldots\right]$$

$$x(2x+1)\left[a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \ldots\right]$$

$$- a_0x^m - a_1x^{m+1} - a_2x^{m+2} - a_3x^{m+3} \ldots$$

$$= 2a_0m(m-1)x^m + 2a_1m(m+1)x^{m+1} + 2a_2(m+2)(m+3)x^{m+2} + 2a_3(m+3)(m+2)x^{m+3} + \ldots$$

$$+ (2x^2+x)\left[a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \ldots\right]$$

$$- a_0x^m - a_1x^{m+1} - a_2\cancel{(m+2)}x^{m+2} - a_3x^{m+3} \ldots$$

$$= 2a_0m(m-1)x^m + 2a_1m(m+1)x^{m+1} + 2a_2(m+2)(m+3)x^{m+2} + \ldots$$

$+ 2a_0$

when roots of the indicial eq^n are equal:

$$y = c_1(y_1)_m + c_2\left(\frac{\partial y_*}{\partial m}\right)_m$$ $\Big\}$ y_* : in terms of m
y_1 : placing $m=0$.

Q. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$

$P_0(x) = x$

$P_0(0) = 0$

$\therefore$ 0 is a singular point

$Q_1(x) = (x-x_0)P_0(x) = (x-x_0)\cdot\frac{P_1(x)}{P_0(x)} = x\cdot\frac{1}{x} = 1 \neq \infty$

$Q_2(x) = (x-x_0)^2 q(x) = x^2\cdot\frac{x}{x} = 0. \neq \infty$ $\Big\}$ $x=0$ is Regular singular.

$\therefore$ series solution is assumed as:

$$y = x^m[a_0 + a_1x + a_2x^2 + \cdots a_nx^n]$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \cdots a_nx^{n+1}$$

$$\frac{dy}{dx} = ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + a_4(m+4)x^{m+3} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + a_3(m+2)(m+3)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \cdots$$

$$\therefore x[m(m-1)a_0x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + a_3(m+2)(m+3)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \cdots]$$
$$+ ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + a_4(m+4)x^{m+3} + \cdots$$
$$+ a_0x^{m+1} + a_1x^{m+2} + a_2x^{m+3} + a_3x^{m+4} + \cdots$$

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$= x^{m-1}\left(a_0 m(m-1) + m a_0\right) + x^m\left(a_1 m(m+1) + a_1(m+1)\right) +$

$x^{m+1}\left(a_2(m+1)(m+2) + a_2(m+2) + a_0\right) + x^{m+2}$

$\left(a_3(m+2)(m+3) + a_3(m+3) + a_1\right) + x^{m+3}.$

$\left(a_4(m+3)(m+4) + a_4(m+4) + a_2\right) +$

To find the value of m, a

$a_0\left[m(m-1) + m\right] = 0$

$a_0 \neq 0$

$\therefore m\left[m-1+1\right] = 0 \Rightarrow m = 0, 0$ (equal roots)

coefficient of $x^m = 0$

$a_1\left[m(m+1) + (m+1)\right] = 0$

$a_1\left[(m+1)^2\right] = 0$

$\boxed{a_1 = 0}$ $(\because m = 0)$

coefficient of $x^{m+1} = 0$.

$a_2\left[(m+1)(m+2) + (m+2)\right] + a_0 = 0$

$a_2\left[2+2\right] = -a_0$ (placing $m=0$)

$\boxed{a_2 = -\frac{a_0}{4}}$

coefficient of $x^{m+2} = 0$

$a_3\left[(m+2)(m+3) + m+3 + 0\right] = 0$

$a_3\left[6+3\right] = 0$

$\boxed{a_3 = 0}$

coefficient of $x^{m+3} = 0$

$$\cancel{a_4\left[(m+3)(m+4)+m+4-\frac{a_0}{4}\right]=0}$$

$$\cancel{a_4\left[12+4-\frac{a_0}{4}\right]=0}$$

$$\cancel{16a_4-\frac{a_0a_4}{4}=0}$$

$$\cancel{16\cancel{a_4}=\frac{a_0\cancel{a_4}}{4}}$$

$$\cancel{\boxed{a_4=0}}$$

$$a_4\left[(m+3)(m+4)+(m+4)\right]-\frac{a_0}{4}=0$$

$$a_4(12+4)=\frac{a_0}{4}$$

$$\boxed{a_4=\frac{a_0}{64}.}$$

$$y_1=\left(a_0+0-\frac{a_0x^2}{4}+0+\frac{a_0x^4}{64}-\cdots\right)\cdot x^0$$

$$y_1=a_0-\frac{a_0x^2}{4}+\frac{a_0x^4}{64}-\cdots$$

$\therefore$ $\cancel{y} = a_0$

$$a_2=\frac{-a_0}{(m+1)(m+2)+(m+2)}$$

$$=\frac{-a_0}{(m+2)[m+1+1]}=\frac{-a_0}{(m+2)^2}$$

$a_3=0$

$$a_4=\frac{-a_2}{(m+3)(m+4)+(m+4)}=\frac{-a_2}{(m+4)(m+4)}$$

$$=\frac{-a_2}{(m+4)^2}=\frac{-(-a_0)}{(m+2)^2(m+4)^2}$$

$$=\frac{a_0}{(m+2)^2(m+4)^2}.$$

$$\therefore y_* = x^m\left[a_0+a_1x+a_2x^2+a_3x^4+\cdots\right]$$

$$y_* = x^m\left[a_0-\frac{a_0x^2}{(m+2)^2}+\frac{a_0x^4}{(m+2)^2(m+4)^2}\cdots\right]$$

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$$\frac{\partial y_*}{\partial m} = x^m \log x \left[a_0 - \frac{a_0 x^2}{(m+2)^2} + \frac{a_0 x^4}{(m+2)^2 (m+4)^2} - - - - - \right] +$$

$\therefore\ y = c$

$$x^m \cdot \frac{\partial}{\partial m} \left[a_0 - \frac{a_0 x^2}{(m+2)^2} + \frac{a_0 x^4}{(m+2)^2(m+4)^2} - - - - - \right]$$

$$= x^m \log x \left[a_0 - \frac{a_0 x^2}{(m+2)^2} + \frac{a_0 x^4}{(m+2)^2(m+4)^2} - - - - - - \right] +$$

$$x^m \left[0 - a_0(-2)(m+2)^{-3} x^2 + a_0 \frac{\partial}{\partial m} (m+2)^{-2}(m+4)^{-2} x^4 \cdots \right]$$

Roots

$$= x^m \log x \left[a_0 - \frac{a_0 x^2}{(m+2)^2} + \frac{a_0 x^4}{(m+2)^2(m+4)^2} - - - - \right] +$$

$$x^m \left[0 + \frac{2a_0 x^2}{(m+2)^3} + a_0 \cdot \left\{ (m+4)^{-2}(-2)(m+2)^{-3} + (m+2)^{-2}(-2)(m+4)^{-3} \right\} x^4 + - - - - \right]$$

$Q\ x$

$$= x^m \log x \left[a_0 - \frac{a_0 x^2}{(m+2)^2} + \frac{a_0 x^4}{(m+2)^2(m+4)^2} - - - \right] + x^m \left[\frac{2a_0 x^2}{(m+2)^3} + \right.$$

$$\left. a_0 \left\{ \frac{-2x^4}{(m+2)^3(m+4)^2} - \frac{2x^4}{(m+2)^2(m+4)^3} \right\} - - - - \right]$$

$Q_1(x$

$$y = c_1 y_1 + c_2 \left(\frac{\partial y_*}{\partial m} \right)_{m=0}$$

$$= \cancel{c_1 \left[a_0 - \frac{a_0 x^2}{4} + \frac{a_0 x^4}{64} \right]} + c_2 \left[a_0 x^m \cancel{\log x} - \cancel{\frac{x^m \log x \cdot a_0 x^2}{(m+2)^2}} \right.$$

$$\left. \frac{\partial y_*}{\partial m} \right|_{m=0} = \log x \left(a_0 - \frac{a_0 x^2}{2^2} + \frac{a_0 x^4}{2^2 \cdot 4^2} - - - \right) + \frac{2a_0 x^2}{2^3} - 2a_0 x^4 \left(\frac{1}{2^3 \cdot 4^2} + \frac{1}{2^2 \cdot 4^3} \right)$$

$$\therefore\ y = c_1\left[a_0 - \frac{a_0x^2}{4} + \frac{a_0x^4}{64} - - -\right] + c_2\left\{\log x\left[a_0 - \frac{a_0x^2}{2^2} - - -\right]\right.$$

$$\left. + \frac{2a_0x^2}{2^3} - 2a_0x^4\left(\frac{1}{2^3.4^2} + \frac{1}{2^2.4^3} - - -\right)\right.$$

<u>Roots are distinct & differ by an integer:</u>

$$y = c_1(y)_{m_2} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1} \quad ; m_1 < m_2.$$

$$a_0 = b_0(m - m_1)$$

Q $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$

$P_0(x) = x(1-x)$

$P_0(0) = 0$

$\therefore$ singular point

$Q_1(x) = (x - x_0)\,P(x)$

$= (x-0)\dfrac{(-1-3x)}{x(1-x)}$

$= \dfrac{-\cancel{x}(1+3x)}{\cancel{x}(1-x)}$

$= -1 \quad \neq \infty$

$Q_2(x) = (x - x_0)^2 . q(x)$

$= (x-0)^2 . \dfrac{(-1)}{x(1-x)}$

$= \dfrac{x^{\cancel{2}}}{(\cancel{x})(1-x)} = \dfrac{x}{x-1} = 0. \quad \neq \infty$

$\left.\frac{1}{2^3.4^3}\right)$

$\therefore$ Regular singular point.

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$$y = x^m[a_0 + a_1x + a_2x^2 + \cdots a_nx^n]$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + a_4x^{m+4} + \cdots$$

$$\frac{dy}{dx} = ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} +$$

$$a_4(m+4)x^{m+3} + \cdots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m +$$

$$a_3(m+2)(m+3)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \cdots$$

$$\therefore \quad x\frac{d^2y}{dx^2} - x^2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3x\frac{dy}{dx} - y = 0$$

$$\Big[m(m-1)a_0x^{m-1} + a_1m(m+1)x^m + a_2(m+1)(m+2)x^{m+1} + a_3(m+2)(m+3)x^{m+2} + a_4(m+3)(m+4)x^{m+3} + \cdots\Big] +$$

$$\Big[-m(m-1)a_0x^m - a_1m(m+1)x^{m+1} - a_2(m+1)(m+2)x^{m+2} - a_3(m+2)(m+3)x^{m+3} - \cdots\Big] +$$

$$\Big[-ma_0x^{m-1} - a_1(m+1)x^m - a_2(m+2)x^{m+1} - a_3(m+3)x^{m+2} - a_4(m+4)x^{m+3} \cdots\Big]$$

$$+3\Big[-ma_0x^{m} - a_1(m+1)x^{m+1} - a_2(m+2)x^{m+2} - a_3(m+3)x^{m+3} - a_4(m+4)x^{m+4} \cdots\Big]$$

$$-a_0x^m - a_1x^{m+1} - a_2x^{m+2} - a_3x^{m+3} - a_4x^{m+4} \cdots$$

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$$x^{m-1}\left[a_0 m(m-1) - m a_0 - \cancel{3m a_0}\right] + x^m\left[a_1 m(m+1) - m(m-1)a_0\right.$$

$$\left. - a_1(m+1) - \cancel{3a_1(m+1)} - a_0\right]$$

$3ma_0$ $\qquad -3a_1(m+1)$

$$+x^{m+1}\left[a_2(m+1)(m+2) - a_1 m(m+1) - a_2(m+2) - \cancel{3a_2(m+2)}\right.$$

$$\left. - a_1\right]$$

$$+x^{m+2}$$

$$a_0 m(m-1) - m a_0 - \cancel{3m a_0} = 0$$

$$a_0\left[m(m-1) - 4m\right] = 0$$

$$a_0 \neq 0$$

$$\therefore \boxed{m=0} \text{ or } \quad m-1-1=0$$

$$\cancel{(m-1)-4=0} \qquad \boxed{m=2}$$

$$a_1 m(m+1) - m(m-1)a_0 = 0$$

$$a_1 = \frac{\cancel{m}(m-1)a_0}{\cancel{m}(m+1)} \Rightarrow \boxed{\cancel{a_1 = \left(\frac{m-1}{m+1}\right)a_0}}$$

$$a_2(m+1)(m+2) - a_1 m(m+1)$$

$$a_1 m(m+1) - m(m-1)a_0 - a_1(m+1) - \overset{3}{4}ma_0 - a_0.$$

$$a_1\left[m(m+1) - (m+1)\right] - a_0\left[m(m-1) - 4m\right] = 0$$

$$\cancel{a_1 = \frac{a_0\left[m(m-1-4)\right]}{(m+1)(m-1)}}$$

$$\cancel{a_1 = \frac{a_0 m(m-5)}{(m+1)(m-1)}}$$

Legendre's differential Equation:

$$(1-x^2)\,y'' - 2xy' + (n)(n+1)y = 0$$

n: Non-negative integer.

$x=0$: ordinary point, $\because P_0(0)=1$

$x=\pm 1$: singular point, $\because P_0(1)=0$.

We'll find solution for ordinary point only.

$\therefore\ y = \Sigma a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad a_n x^n$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$\therefore\ (1-x^2)\left[2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots\right]$

$-2x\left[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots\right] + n(n+1)$

$\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]$

$(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots) - (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4$
$+ 20a_5 x^5 + 30a_6 x^6 \cdots$

$-(2a_1 x + 4a_2 x^2 + 6a_3 x^3 + 8a_4 x^4 + 10a_5 x^5 + \cdots)$

$+n(n+1)a_0 + n(n+1)a_1 x + n(n+1)x^2 \cdot a_2 + n(n+1)a_3 \cdot x^3 + \cdots$

$$2a_2 + n(n+1)a_0 + x\left[6a_3 - 2a_1 + n(n+1)a_1\right] + x^2\left[12a_4 - 2a_2 - 4a_2 + n(n+1)a_2\right]$$

$+ \ldots$

$2a_2 + n(n+1)a_0 = 0$

$$\Rightarrow \boxed{a_2 = -\frac{(n+1)\cdot n a_0}{2}}$$

$6a_3 - 2a_1 + (n+1)na_1 = 0$

$a_1\cdot\left[-2 + n(n+1)\right] = -6a_3$

$a_3 = -\frac{a_1}{6}\left(-2 + n(n+1)\right)$

$$\boxed{a_3 = \frac{a_1\left[2 - n(n+1)\right]}{6}}$$

$\Downarrow$

$a_3 = -\frac{a_1}{6}\left[n^2 + n - 2\right]$

$= -\frac{a_1}{6}(n-1)(n+2)$.

$12a_4 + a_2\left[-2-4 + n(n+1)\right] = 0$

$a_4 = \frac{\left[6 - n(n+1)\right]a_2}{12}$

$= -\frac{\left[6 - n(n+1)\right]n(n+1)\cdot a_0}{24}$.

$$\therefore y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 \ldots\right]$$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}\cdot x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 \ldots\right]$$

Legendre's function of first kind:

$$P_n(x) = \frac{1\cdot3\cdot5\cdot7\cdot\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2\cdot4\cdots(2n-1)(2n-3)}\right]$$

$$a_0 = \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{n!}$$

$P_n(x)$:

It is a terminating series & gives what is called Legendre's polynomial for different values of 'n' such that $P_n(1)=1$.

$$P_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}$$

Again when 'n' is a positive integer,

$$a_0 = \frac{n!}{1\cdot3\cdot5\cdots(2n-1)}$$

$$Q_n(x) = \frac{n!}{1\cdot3\cdot5\cdots(2n-1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+5)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-n-5}\cdots\right]$$

Non-terminating

$y = c_1 P_n(x) + c_2 Q_n(x)$: general solution of Legendre's equation.

Rodrigue's Formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n (x^2-1)^n}{dx^n} \quad ; \quad v = (x^2-1)^n$$

Lebnitz theorem:

$$D^n(uv) = (D^n u)v + {}^nC_1 (D^{n-1}u)(Dv) + {}^nC_2 (D^{n-2}u)(D^2v)$$

$$(1-x^2)\frac{d^{n+1}}{dx^{n+1}}\cdot v_1$$

$$v_1 = \frac{dv}{dx} = n(x^2-1)^{n-1}\cdot 2x$$

$$\Rightarrow v_1(x^2-1) = 2nxv$$

$$(1-x^2)v_1 + 2nxv = 0 \quad \text{—(1)}$$

Differentiating (1) $(n+1)$ times, ${}^{n+1}C_2$

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{n(n+1)}{2!}(-2)v_n$$

$$+ 2n\left[x v_{n+1} + (n+1)v_n\right] = 0$$

$$(1-x^2)v_{n+2} - 2x v_{n+1} + n(n+1)\overset{v_n}{y} = 0$$

$$\boxed{(1-x^2)y'' - 2x\cdot y' + n(n+1)y = 0}$$: legendre's equation

$\{$with $y = v_n\}$

But the solution of legendre's equations are $P_n(x)$ & $Q_n(x)$

$$v_n = \frac{d^n}{dx^n}(x^2-1)^n$$

containing positive powers of x, it must be a constant multiple of $P_n(x)$.

$$v_n = c\cdot P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n.$$

$$\frac{d^n}{dx^n}\left[\underset{u}{(x+1)^n}\,\underset{v}{(x-1)^n}\right]$$

$$= \frac{d^n (x+1)^n}{dx^n}\cdot(x-1)^n + {}^nC_1\,\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\left(n(x-1)^{n-1}\right) + {}^nC_2$$

$$\frac{d^{n-2}}{x^{n-2}}(x+1)^n\cdot n(n-1)(x-1)^{n-2}\;.....$$

$$.... (x+1)^n\cdot\frac{d^n}{dx^n}(x-1)^n.$$

$\because \frac{d^n}{dx^n}(x-1)^n = n!$

$\therefore CP_n(x) = n!(x+1)^n$ + terms containing powers of $(x-1)$

when $x=1$,

$P_n(1) = n!\,2^n$

$\therefore C = n!\,2^n$

$\therefore CP_n(x) = \cancel{2^n \cdot n!}\ \frac{d^n(x^2-1)^n}{dx^n}$

$$\boxed{P_n(x) = \frac{1}{\underbrace{2^n n!}_{C}} \frac{d^n(x^2-1)^n}{dx^n}}$$

Substituting values of n as $0, 1, 2 \ldots$

$\therefore$ for $n=0$,

$P_0(x) = \frac{1}{2^0 \cdot 1!} \frac{d^0(x^2-1)^0}{dx^0} = 1$

$P_1(x) = \frac{1}{2} \frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}} \cdot \cancel{2}x = x$

$P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2(x^2-1)^2}{dx^2} = \frac{1}{8} \frac{d^2}{dx^2}(x^4+1-2x^2)$

$= \frac{1}{8} \cdot \frac{d}{dx} = \frac{1}{8}\left[4x^3 - 4x\right]$

$= \frac{1}{8} \cdot (12x^2 - 4)$

$P_2(x) = \frac{3x^2-1}{2}$

$$P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3}{dx^3} (x^2-1)^3$$

$$\frac{1}{48} \cdot \frac{d^3}{dx^3}\left[x^6 - 1 - 3x^2(x^2-1)\right]$$

$$\frac{1}{48} \frac{d^3}{dx^3}\left[x^6 - 1 - 3x^4 + 3x^2\right]$$

$$\frac{1}{48} \cdot \frac{d^2}{dx^2}\left[6x^5 - 12x^3 + 6x\right]$$

$$\frac{1}{48} \cdot \frac{d}{dx}\left[30x^4 - 36x^2 + 6\right]$$

$$\frac{1}{48}\left[120x^3 - 72x\right] = \frac{10x^3 - 6x}{4} = \frac{5x^3 - 3x}{2}$$

$$P_4(x) = \frac{1}{2^4 \cdot 4!} \frac{d^4}{dx^4} (x^2-1)^4$$

$$\frac{1}{16 \cdot 24} \cdot \frac{d^4}{dx^4}\Big[{}^4C_0 \cdot (x^2)^4 (-1)^0 + {}^4C_1 \cdot (x^2)^3 \cdot (-1)^1 + {}^4C_2 \cdot (x^2)^2 \cdot (-1)^2$$
$$+ {}^4C_3 \cdot (x^2)^1 (-1)^3 + {}^4C_4 \cdot (x^2)^0 (-1)^4\Big]$$

$$\frac{1}{16 \cdot 24} \cdot \frac{d^4}{dx^4}\left[x^8 - 4 \cdot x^6 + 6x^4 - 4x^2 + 1\right]$$

$$\frac{1}{16 \cdot 24} \frac{d^3}{dx^3}\left[8x^7 - 24x^5 + 24x^3 - 8x\right]$$

$$\frac{1}{16 \cdot 24} \cdot \frac{d^2}{dx^2}\left[56x^6 - 120x^4 + 72x^2 - 8\right]$$

$$\frac{1}{16 \cdot 24} \frac{d}{dx}\left[56 \cdot 6x^5 - 480x^3 + 144x\right]$$

$$\frac{1}{16 \cdot 24}\left[56 \cdot 6 \cdot 5x^4 - 480 \cdot 3 \cdot x^2 + 144\right] = \frac{\cancel{56}^{\,\cancel{14}\,7} \cdot 5x^4 - \cancel{80}^{\,\cancel{20}\,10} \cdot 3x^2 + \cancel{24}^{\,\cancel{6}\,3}}{\cancel{16} \cdot \cancel{4}\ 8}$$

$P_4(x) = \dfrac{35x^4 - 30x^2 + 3}{8}$

$\cancel{P_5(x) =}$

Q Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of legendre's polynomial.

$x^4 + 3x^3 - x^2 + 5x - 2 = aP_4(x) + bP_3(x) + c \cdot P_2(x) + dP_1(x) + eP_0(x)$

Placing $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ & $P_4(x)$ we get

$= \dfrac{a}{8} \cdot \left[35x^4 - 30x^2 + 3\right] + \dfrac{b}{2} \cdot \left[5x^3 - 3x\right] + \dfrac{c}{2}\left[3x^2 - 1\right] + d[x] + e[1]$

$= \dfrac{35a}{8} x^4 + x^3\left[\dfrac{5b}{2}\right] + x^2\left[\dfrac{-30a}{8} + \dfrac{3c}{2}\right] + x\left[\dfrac{-3b}{2} + d\right]$

$+ \dfrac{3a}{8} - \dfrac{c}{2} + e.$

$\therefore$ comparing both sides,

$\dfrac{35a}{8} = 1 \Rightarrow \boxed{a = \dfrac{8}{35}}$

$\dfrac{5b}{2} = 3 \Rightarrow \boxed{b = \dfrac{6}{5}}$

$\dfrac{-30a}{8} + \dfrac{3c}{2} = -1$

$-\dfrac{\cancel{30}^{6}}{\cancel{8}} \cdot \dfrac{\cancel{8}}{\cancel{35}_{7}} + \dfrac{3c}{2} = -1$

$\dfrac{3c}{2} = \dfrac{6}{7} - 1$

$\dfrac{3c}{2} = \dfrac{-1}{7} \Rightarrow \boxed{c = \dfrac{-2}{21}}$

$\dfrac{-3b}{2} + d = 5$

$-\dfrac{3}{\cancel{2}} \cdot \dfrac{\cancel{6}^{3}}{5} + d = 5$

$d = \dfrac{9}{5} + 5$

$\boxed{d = \dfrac{34}{5}}$

$\dfrac{3a}{8} - \dfrac{c}{2} + e = -2$

$\dfrac{3}{\cancel{8}} \cdot \dfrac{\cancel{8}}{35} + \dfrac{1}{\cancel{2}} \cdot \dfrac{\cancel{2}}{21} + e = -2$

$\dfrac{3}{35} + \dfrac{1}{21} + e = -2$

$$\frac{38}{35} + e = -2$$

$$e = -2 - \frac{38}{35}$$

$$e = \frac{-70-38}{35}$$

$$\boxed{e = \frac{-108}{35}}$$

$$\frac{63+35}{35 \cdot 21} + e + 2 = 0$$

$$\boxed{e = \frac{-224}{105}}$$

Generating function of legendre's polynomial

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$(1-2xh+h^2)^{-1/2} = \{1 - h(2x-h)\}^{-1/2}$$

Expand using Binomial Theorem & equate coefficient of h^n to get $P_n(x)$.

Recurrence formula:

I. $(2n+1)xP_n = (n+1)P_{n+1} + n \cdot P_{n-1}$

II. $nP_n = xP_n' - P_{n-1}'$

III. $(2n+1)P_n = P_{n+1}' - P_{n-1}'$

IV. $(n+1)P_n = P_{n+1}' - xP_n'$

V. $(1-x^2)P_n' = n(P_{n-1} - xP_n)$

VI. $(1-x^2)P_n' = (n+1)(xP_n - P_{n+1})$

I.

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{--- (1)}$$

Differentiating both sides wrt h,

$$\Rightarrow -\frac{1}{2}(1-2xh+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} n \cdot h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-3/2} = \sum n \cdot h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2}(1-2xh+h^2)^{-1} = \sum n \cdot h^{n-1} P_n(x)$$

$$\Rightarrow (x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum n \cdot h^{n-1} P_n(x)$$

$$(x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2hx+h^2)\sum_{n=0}^{\infty} n \cdot h^{n-1} P_n(x)$$

(from (1))

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$$(x-h)\left[h^0 P_0(x)+hP_1(x)+h^2P_2(x)+\cdots h^{n-1}P_{n-1}(x)+h^nP_n(x)\right] =$$

$$(1-2xh+h^2)\left[P_1(x)+2hP_2(x)+\cdots (n-1)h^{n-2}P_{n-1}(x)+ n\cdot h^{n-1}P_n(x)+(n+1)h^nP_{n+1}(x)\right]$$

Equating the coefficients of h^n on both sides,

$$xP_n(x)-P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) \cancel{-2x(n+1)P_{n+1}} + (n-1)\cdot P_{n-1}(x)$$

$$\Rightarrow \boxed{(2n+1)xP_n = (n+1)P_{n+1} + n\cdot P_{n-1}(x)}$$

11/2/19.

Q.1. $x^2 \cdot y'' + x(x-1)y' + (1-x)y = 0.$

$P_0(0) = 0.$

$\therefore 0$ is a singular point.

$Q_1(x) = (x-0) \cdot P(x)$
$= x \cdot \frac{\cancel{x}(x-1)}{x^2} = 0-1 = -1$

$\neq \infty$

$Q_2 x = (x-0)^2 \cdot \frac{(1-x)}{x^2} = 1$

$\neq \infty.$

$\therefore$ '0' is regular singular.

$\therefore$ series solution is $y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$

$\therefore y = x^m a_0 + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + a_5 x^{m+5} + \cdots$

$\frac{dy}{dx} = m \cdot a_0 x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2}$
$+ a_4(m+4)x^{m+3} + a_5(m+5)x^{m+4} + \cdots$

$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$
$+ a_3(m+2)(m+3)x^{m+1} + a_4(m+3)(m+4)x^{m+2}$
$+ a_5(m+4)(m+5)x^{m+3} + \cdots$

$\therefore$

$[a_0 m(m-1)x^m + a_1 m(m+1)x^{m+1} + a_2(m+1)(m+2)x^{m+2} + a_3(m+2)(m+3)x^{m+3} + a_4(m+3)(m+4)x^{m+4} \cdots]$

$+ (x^2 - x)[m a_0 x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + a_4(m+4)x^{m+3} + \cdots]$

$+ (1-x)[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \cdots]$

$$2\left[a_0 m(m-1)x^m + a_1 m(m+1)x^{m+1} + a_2(m+1)(m+2)x^{m+2} + a_3(m+2)(m+3)x^{m+3} \ldots\ldots\right]$$

$$-2\left[m a_0 x^{m-1} + (m+1)a_1 x^m + a_2(m+2)x^{m+1}\right.$$

$$a_0 m(m-1)x^m + a_1 m(m+1)x^{m+1} + a_2(m+1)(m+2)x^{m+2} + a_3(m+2)(m+3)x^{m+3} + \ldots$$

$$m a_0 x^{m+1} + (m+1)a_1 x^{m+2} + a_2(m+2)x^{m+3} + a_3(m+3)x^{m+4} + \ldots$$

$$-\left[m a_0 x^m + (m+1)a_1 x^{m+1} + (m+2)a_2 x^{m+2} + (m+3)a_3 x^{m+3} \ldots\right]$$

$$+a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots$$

$$-\left[a_0 x^{m+1} + a_1 x^{m+2} + a_2 x^{m+3} + a_3 x^{m+4} + \ldots\right].$$

$$x^m\left[a_0 m(m-1) - m a_0 + a_0\right] + x^{m+1}\left[a_1 m(m+1) + m a_0 - a_1(m+1) + a_1 - a_0\right]$$

$$+x^{m+2}\left[a_2(m+1)(m+2) + a_1(m+1) - (m+2)a_2 + a_2 - a_1\right]$$

$$a_0\left[m(m-1) - m + 1\right] = 0$$

$\because a_0 \neq 0$

$\therefore (m-1)^2 = 0$

$m = 1, 1.$

$\because$ Roots are equal,

$$\therefore y = C_1(y_1)_m + C_2\left(\frac{\partial y_1}{\partial m}\right)_m.$$

$y = x^m [a_0 + a_1 x + a_2 x^2 + \cdots]$

$2\cancel{a_1} + \cancel{a_0} - \cancel{2a_1} + a_1 - \cancel{a_0} = 0$

$\boxed{a_1 = 0}$

$6a_2 + 2a_1 - 3a_2 + a_2 - a_1 = 0$

$\boxed{a_2 = 0}$

$y_1 = a_0 x\, x^m [a_0 + a_1 x + \cdots]$

$\boxed{y_1 = a_0 x}$

~~$y = a_0 x^m$~~

$y = x^m [a_0 + a_1 x + a_2 x^2 + \cdots]$

$\frac{\partial y}{\partial m} = x^m \log x [a_0 + a_1 x + a_2 x^2 + \cdots] + x^m \cdot 0$

$= x^m \log x [a_0 + a_1 x + a_2 x^2 \cdots] \Big|_{m=1}$

$= x \log x [a_0 + a_1 x + a_2 x^2 + \cdots]$

$a_1 [m(m+1) - (m+1) + 1] + a_0 [m-1] = 0$

$a_1 = \frac{(m-1)}{m^2 + \cancel{m} - \cancel{m} - \cancel{1} + \cancel{1}} = \frac{-m-1}{m^2}$

$a_2 [(m+1)(m+2) - (m+2) + 1] + a_1 [(m+1) - 1] = 0$

$a_2 [(m+1)(m+2) - (m+2) + 1] = -a_1 [m + \cancel{1} - \cancel{1}]$

$a_2 [(m+2)(m + \cancel{1} - \cancel{1}) + 1] = -a_1 m$

$a_2 = \frac{-a_1 m}{m^2 + 2m + 1} = -a_1 \cdot \frac{m}{(m+1)^2} = +(m+1) \cdot \frac{m}{m^2 (m+1)^2}$

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$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$$

$$y = a_0 x^m - \frac{(m-1)}{m^2} x^{m+1} + \frac{m(m-1)}{m^2(m+1)^2} \cdot x^{m+2} + \cdots\cdots$$

$$\frac{\partial y}{\partial m} = a_0 \cdot x^m \log x - \frac{\partial}{\partial m}\left[\frac{(m-1)x^{m+1}}{m^2}\right] + \frac{\partial}{\partial m}\left[\frac{m(m-1)\, x^{m+2}}{m^2(m+1)^2}\right]$$

$$-\frac{\partial}{\partial m}\left[\frac{\cancel{m} \cdot x^{m+1}}{m^{\cancel{2}}} - \frac{x^{m+1}}{m^2}\right]$$

$$= \left\{\left(\frac{-1}{m^2} \cdot x^{m+1}\right) + \left(x^{m+1} \log x\right) \cdot \frac{1}{m}\right.$$

$$\left. - \left[\frac{-2}{m^3} \cdot x^{m+1} + \frac{1}{m^2} \cdot x^{m+1} \log x\right]\right\}$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x - \left\{\left[\frac{-x^{m+1}}{m^2} + \frac{x^{m+1} \log x}{m}\right] - \left[\frac{-2x^{m+1}}{m^3} + \frac{x^{m+1}\log x}{m^2}\right]\right\}$$

$$y = a_0 x^m \left[a_0 + a_1 - \frac{(m-1)}{m^2} \cdot x^{m+1} + \frac{m(m-1)}{m^2(m+1)^2} x^{m+2} + \cdots\right]$$

$$\frac{\partial y}{\partial m} = x^m \log x \left[a_0 - \frac{(m-1)}{m^2} x^{m+1} + \frac{m(m-1)}{m^2(m+1)^2} x^{m+2} + \cdots\right]$$

$$+ x^m \cdot \left[0 - \right.$$

$$\frac{\partial}{\partial m}\left[\frac{\cancel{m}(m-1)x^{m+2}}{m^{\cancel{2}}(m+1)^2}\right]$$

$$\frac{\partial}{\partial m}\left[\frac{(m+1)\, x^{m+2}}{m(m+1)^2} - \frac{x^{m+2}}{m(m+1)^2}\right]$$

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$$= x^{m+2}\log x + x^{m+2}\cdot\left[\frac{\partial}{\partial m}\left(\frac{m+1}{m^3+2m^2+m}\right)\right]$$

$$-\left[x^{m+2}\log x + x^{m+2}\cdot\frac{\partial}{\partial m}\left(\frac{1}{m^3+2m^2+m}\right)\right]$$

$$= x^{m+2}\log x + x^{m+2}\left[\frac{(m^3+2m^2+m)-(3m^2+4m+1)}{(m^3+2m^2+m)^2}\right]$$

$$-\left[x^{m+2}\log x + x^{m+2}\cdot\left(\frac{-3m^2-4m-1}{(m^3+2m^2+m)^2}\right)\right]$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x - \left\{\left[\frac{-x^{m+1}}{m^2} + \frac{x^{m+1}\log x}{m}\right] - \left[\frac{-2x^{m+1}}{m^3} + \frac{x^{m+1}\log x}{m^2}\right]\right\}$$

$$+ x^{m+2}\log x + x^{m+2}\left[\frac{(m^3+2m^2+m)-(3m^2+4m+1)}{(m^3+2m^2+m)^2}\right]$$

$$-\left[x^{m+2}\log x + x^{m+2}\left(\frac{-3m^2-4m-1}{(m^3+2m^2+m)^2}\right)\right] + \cdots$$

$$\left.\frac{\partial y}{\partial m}\right|_{m=1} = a_0 x\log x - \left\{\left[\frac{-x^2}{1} + \frac{x^2\log x}{1}\right] - \left[\frac{-2x^2}{1} + \frac{x^2\log x}{1}\right]\right\}$$

$$+ x^3\log x + x^3\left[\frac{(1+2+1)-(3+4+1)}{(1+2+1)^2}\right] - \left[x^3\log x\right.$$

$$\left. + x^3\left(\frac{-3-4-1}{(1+2+1)^2}\right)\right]$$

$$= a_0 x\log x - \left\{(-x^2+x^2\log x) - (-2x^2+x^2\log x)\right\}$$

$$+ x^3\log x + x^3\left[\frac{4-8}{16}\right] - \left[x^3\log x + x^3\left(\frac{-8}{16}\right)\right]$$

$$= a_0 x\log x + x^2 - x^2\log x - 2x^2 - x^2\log x + x^3\log x - \frac{1}{4}x^3 - x^3\log x$$

$$+ x^3/2 + \cdots$$

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$a_0 \log x + x^2(-1-2\log x) + x^3\left(\cancel{\log x} - \frac{1}{4} - \cancel{\log x} + \frac{1}{2}\right) + \cdots$

~~$x a_0 \log x + x^2(-1-$~~

$a_0\, x\log x - x^2(1+2\log x) + \frac{x^3}{4} + \cdots$

$$y_2 = x\log x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n!\, n}$$

II. $n \cdot P_n = x P_n' - P_{n-1}'$

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{—} (1)$$

Differentiating wrt x,

$$-\frac{1}{\cancel{2}}(1-2hx+h^2)^{-3/2}(-\cancel{2}h) = \sum_{n=0}^{\infty} h^n P'_n(x)$$

$$h(1-2hx+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P'_n(x) \quad \text{—} (2)$$

Differentiating (1) wrt h,

$$-\frac{1}{2}(1-2hx+h^2)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} n \cdot h^{n-1} P_n(x)$$

$$(1-2hx+h^2)^{-3/2}(x-h) = \sum_{n=0}^{\infty} n \cdot h^{n-1} P_n(x) \quad \text{—} (3)$$

Multiplying (2) with $(x-h)$,

$$h(x-h)(1-2hx+h^2)^{-3/2} = (x-h) \sum_{n=0}^{\infty} h^n P_n'(x)$$

from (3),

$$h \cdot \sum_{n=0}^{\infty} n \cdot h^{n-1} P_n(x) = (x-h) \cdot \sum_{n=0}^{\infty} h^n P_n'(x)$$

$$h \cdot \left[P_1(x) + 2h P_2(x) + 3h^2 P_3(x) + \cdots (n-1) h^{n-2} P_{n-1}(x) + n \cdot h^{n-1} P_n(x) \right]$$

$$= (x-h) \left[P_0'(x) + h P_1'(x) + h^2 P_2'(x) + \cdots h^{n-1} P_{n-1}'(x) + h^n P_n'(x) \right]$$

$\therefore$ Equating the coefficients of h^n,

$$\boxed{n \cdot P_n(x) = x P_n'(x) - P_{n-1}'(x)}$$

III. $(2n+1) P_n = P_{n+1}' - P_{n-1}'$

From recurrence I,

$$(2n+1) x P_n = (n+1) P_{n+1} + n \cdot P_{n-1}(x) \quad \text{—— (1)}$$

Differentiating w.r.t x,

$$(2n+1) P_n + (2n+1) x \cdot P_n' = (n+1) P_{n+1}'(x) + n \cdot P_{n-1}'(x) \quad \text{—— (2)}$$

From recurrence II,

$$n \cdot P_n(x) = x \cdot P_n'(x) - P_{n-1}'(x) \quad \text{—— (3)}$$

$$\Rightarrow x P_n'(x) = n P_n(x) + P_{n-1}'(x)$$

$$(2n+1) \left[P_n + x P_n'(x) \right] = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

From (3),

$$(2n+1) \left[P_n + n P_n(x) + P_{n-1}'(x) \right] = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

$$(2n+1) P_n + n(2n+1) P_n(x) + (2n+1) P_{n-1}'(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

$$P_n(x)\left[(2n+1)+n(2n+1)\right]+\left[(2n+1)-n\right]P_{n-1}'(x)=P_{n+1}'(x)\cdot(n+1)$$

$$P_n(x)\left[(2n+1)\cancel{(n+1)}\right]+\cancel{(n+1)}P_{n-1}'(x)=\cancel{(n+1)}P_{n+1}'(x)$$

$$\therefore (2n+1)P_n(x)+P_{n-1}'(x)=P_{n+1}'(x)$$

$$\boxed{(2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x)}$$

IV. $(n+1)P_n=(P'_{n+1}-xP_n')$

From recurrence II,

$$nP_n(x)=xP_n'-P_{n-1}'(x)\quad\text{—①}$$

from recurrence III,

$$(2n+1)P_n(x)=P_{n+1}'(x)-P_{n-1}'(x)\quad\text{—②}$$

subtracting ① from ②,

$$(2n+1)P_n(x)-nP_n(x)=P_{n+1}'(x)-\cancel{P_{n-1}'(x)}-xP_n'+\cancel{P_{n-1}'(x)}$$

$$\boxed{(n+1)P_n(x)=P_{n+1}'(x)-xP_n'(x)}$$

V. $(1-x^2)P_n'=n(P_{n-1}-xP_n)$

~~Multiplying recurrence IV by x,~~

$$\cancel{(n+1)xP_n(x)=x\cdot P_{n+1}'(x)-x^2P_n'(x)}$$

from recurrence IV,

$$(n+1)P_n(x)=P_{n+1}'(x)-xP_n'(x)\quad\text{—①}$$

$$n\to n-1$$

$$(n-\cancel{1}+\cancel{1})P_{n-1}(x)=P_{n-\cancel{1}+\cancel{1}}'(x)-xP_{n-1}'(x)$$

n+1)

$n P_{n-1}'(x) = P_n'(x) - x P_{n-1}'(x)$ — (2)

from recurrance II,

$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$ — (3)

Multiplying (3) by x,

$n x P_n(x) = x^2 P_n'(x) - x \cdot P_{n-1}'(x)$ — (4)

(4) − (2)

$$n x P_n(x) - n P_{n-1}'(x) = x^2 P_n'(x) - \cancel{x P_{n-1}'(x)} - P_n'(x) + \cancel{x P_{n-1}'(x)}$$

$$n x P_n(x) - n P_{n-1}'(x) = P_n'(x)(x^2 - 1)$$

$$n P_{n-1}'(x) - n x P_n(x) = (1-x^2) P_n'(x)$$

$$\boxed{n[P_{n-1}'(x) - x P_n(x)] = (1-x^2) P_n'(x)}$$

VI. $(1-x^2) P_n' = (n+1)(x P_n - P_{n+1})$

from (1) recurrence I,

$(2n+1) x P_n = (n+1) P_{n+1} + n \cdot P_{n-1}$

$(n+1+n) x P_n(x) = (n+1) P_{n+1} + n P_{n-1}$

$(n+1) x P_n(x) + n x P_n(x) = (n+1) P_n(x) + n P_{n-1}(x)$

$(n+1)[x P_n(x) - P_n(x)] = n P_{n-1}(x) - n x P_n(x)$ — (A)

from recurrence (5),

$(1-x^2) P_n'(x) = n[P_{n-1}'(x) - x P_n(x)]$ — (B)

∵ RHS of both are same,

$$\boxed{(1-x^2) P_n' = (n+1)(x P_n - P_{n+1})}$$

① Show that $P_n(1) = 1$.

From generating function,

$$(1-2hx+h^2)^{-1/2} = \Sigma h^n P_n(x)$$

Placing $x=1$,

$$(1-2h+h^2)^{-1/2} = \Sigma h^n P_n(1)$$

$$(h-1)^{\cancel{-2} \cdot 1/\cancel{2}} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$(1-h)^{-1} = \Sigma h^n P_n(1)$$

$$1+h+h^2+\cdots h^n = P_0(1)+hP_1(1)+h^2P_2(1) + \cdots h^n P_n(1)$$

$$\therefore P_n(1) = 1$$

2. $P_n(-x) = (-1)^n P_n(x)$

3. $P_n'(1) = \frac{n(n+1)}{2}$

4. $P_n'(-1) = (-1)^{n+1} \frac{n(n+1)}{2}$

5. $\frac{1-z^2}{(1-(2xz)+z^2)^{3/2}} = \Sigma (2n+1) P_n(x) z^n$.

2. Generating function:

$$(1-2hx+h^2)^{-1/2} = \Sigma h^n P_n(x) \quad \text{--- ①}$$

$x \to (-x)$ $\quad h \to -h$

$$(1+2hx+h^2)^{-1/2} = \Sigma h^n P_n(-x) \quad \text{--- ②}$$

$h \to -h$ in (1),

$$(1+2hx+h^2)^{-1/2} = \Sigma (-h)^n P_n(x)$$

$$(1+2hx+h^2)^{-1/2} = \Sigma (-1)^n h^n P_n(x) \quad \text{—— (3)}$$

from (2) & (3),

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

$$P_0(-x) + h P_1(-x) + \cdots h^n P_n(-x) = P_0(x) - hP_1(x) + h^2 P_2(x) \cdots (-1)^n h^n P_n(x)$$

$$\boxed{P_n(-x) = (-1)^n P_n(x)}$$

3. $P_n'(1) = \frac{n(n+1)}{2}$

~~VI.~~ $(1-x^2) P_n'(x) = (n+1)(x P_n(x) - P_{n+1}(x))$

from Legendre's equation,

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) \cdot P_n(x) = 0$$

$x = 1$,

$$-2P_n'(1) + n(n+1) P_n(1) = 0$$

$$-2P_n'(1) + n(n+1)(1) = 0$$

$$\boxed{P_n'(1) = \frac{n(n+1)}{2}}$$

4. $P_n'(-1) = (-1)^{n+1} \frac{n(n+1)}{2}$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

$x \to -x$

$$(1-x^2) P_n''(-x) + 2x P_n'(-x) + n(n+1) P_n(-x) = 0$$

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$x=1$,

$\cancel{2P_n'(-1)} + n(n+1) P_n(-1) = 0$.

$x\,2P_n'(-x) + n(n+1) P_n(-x) = 0$

$x \cdot 2 \cdot (-1)^n P_n'(x) + n(n+1)(-1)^n P_n(x) = 0$

$x=1$

$2(-1)^n \cdot P_n(1) + n(n+1)(-1)^n P_n(1) = 0$

$2(-1)^n + n(n+1)(-1)^n = 0$

$2x P_n'(-x) + n(n+1)(-1)^n P_n(x) = 0$

$x=-1$

$2P_n'(-1) + n(n+1)(-1)^n P_n(1) = 0$

$2P_n'(-1) + n(n+1)(-1)^n = 0$

$$P_n'(-1) = \frac{-n(n+1)(-1)^n}{2}$$

$$= -(-1)^n \frac{n(n+1)}{2} = (-1)^{n+1} \frac{(n+1)n}{2}$$

5. $\dfrac{1-z^2}{(1-(2xz)+z^2)^{3/2}} = \Sigma (2n+1) P_n(x) z^n$

Generating function,

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$(1-2zx+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

Differentiating wrt z,

$$-\frac{1}{2}(1-2zx+z^2)^{-3/2} \cdot (-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n(x)$$

$$(1-2zx+z^2)^{-3/2}\,(x-z) = \sum_{n=0}^{\infty} n\cdot z^{n-1}\, P_n(x)$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$\frac{d}{dx}\,(1-x^2)\,P_m'(x) + n(n+1)y = 0$$

Orthogonality of legendre's eqⁿ :

(i) $$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \quad ; \; m \neq n$$

$$= \frac{2}{2n+1} \quad ; \; m = n.$$

(ii)

consider legendre's differential equation,

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0 \quad \text{—— (1)}$$

let $P_n(x)$ & $P_m(x)$ be the solutions of (1),

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] + n(n+1)P_n = 0 \quad \text{--- (2)}$$

&

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + m(m+1)P_m = 0 \quad \text{--- (3)}$$

Multiplying (2) by P_m & (3) by P_n,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right]\cdot P_m + n(n+1)P_nP_m = 0 \quad \text{--- (4)}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]P_n + m(m+1)P_nP_m = 0 \quad \text{--- (5)}$$

Subtracting (4) & (5),

$$P_m\cdot\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] - P_n\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + P_nP_m\left[n(n+1) - m(m+1)\right] = 0$$

--- (6)

Integrating (6) by taking limits from -1 to 1

$$P_m\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right]dx - P_n\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]dx$$

$$+\left[n(n+1) - m(m+1)\right]\int_{-1}^{1}P_nP_m\,dx = 0$$

$$P_m\left[(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{1} - P_n\left[(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{1}$$

$$+\left[n(n+1) - m(m+1)\right]\int_{-1}^{1}P_n(x)P_m(x)\,dx$$

$$= 0$$

$$0 + 0 = \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0.$$

$$\boxed{\therefore \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0}$$

(ii).

Generating function

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{—} \ (1) \qquad |x| \le 1 ; \ |z| \le 1$$

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^m P_m(x) \quad \text{—} \ (2)$$

multiplying (1) & (2),

$$(1-2xz+z^2)^{-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^{m+n} P_m P_n \quad \text{—} \ (3)$$

Integrating (3) using limits from -1 to 1,

$$\int_{-1}^{1} (1-2xz+z^2)^{-1}\, dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^{1} P_m P_n z^{m+n}\, dx$$

$m=n$;

$$\int_{-1}^{1} \frac{dx}{1-2xz+z^2} = \sum_{n=0}^{\infty} \int_{-1}^{1} [P_n(x)]^2 z^{2n}\, dx$$

$$\frac{1}{-2x} \ln(1-2xz+z^2)$$

$$\frac{2}{z}\left[z - \frac{z^3}{3} + \frac{z^5}{5} \cdots\right] = \sum_{n=0}^{\infty}\int_{-1}^{1}[P_n(x)]^2 z^{2n}$$

$$\sum_{n=0}^{\infty} \frac{2z^{2n}}{2n+1} = \sum_{n=0}^{\infty}\int_{-1}^{1}(P_n(x))^2 \cdot z^{2n}.$$

Equating coefficient of z^{2n},

$$\int_{-1}^{1}[P_n(x)]^2 = \frac{2}{2n+1} \quad ; m=n.$$

8/2/19.

1. Prove:

$$\int_{-1}^{1}(x^2-1)\,P_{n+1}(x)\cdot P_n'(x)\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

using recurrence V,

$$(1-x^2)\,P_n' = n(P_{n-1} - xP_n)$$

$$(x^2-1)\,P_n' = n(xP_n - P_{n-1})$$

Integrating from -1 to 1,

$$\int_{-1}^{1}(x^2-1)\,P_n' = \int_{-1}^{1} n(xP_n - P_{n-1})\,dx$$

$$\int_{-1}^{1}(x^2-1)\,P_{n+1}P_n'\,dx = \int_{-1}^{1} n(xP_n - P_{n-1})\,P_{n+1}\,dx$$

$$\int_{-1}^{1}(x^2-1)\,P_{n+1}P_n'\,dx = \int_{-1}^{1}(P_n \cdot x n_{\downarrow} - nP_{n-1}P_{n+1})\,dx$$

P_{n+1}.

$$\int_{-1}^{1}(x^2-1)P_{n+1}P_n'\,dx = \int_{-1}^{1} n\cdot P_n\,x\,dx\;P_{n+1} - n\underbrace{\int_{-1}^{1} P_{n-1}P_{n+1}\,dx}_{0}$$

$\because n-1 \neq n+1$

$$\int_{-1}^{1}(x^2-1)P_{n+1}P_n'\,dx = n\int_{-1}^{1} xP_n\cdot dx\;P_{n+1}$$

from recurrence I_1

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

$$\Rightarrow xP_n = \frac{(n+1)P_{n+1} + nP_{n-1}}{(2n+1)}$$

$$\int_{-1}^{1}(x^2-1)P_{n+1}P_n'\,dx = n\int_{-1}^{1}\frac{(n+1)P_{n+1} + nP_{n-1}}{(2n+1)}\,dx\cdot P_{n+1}$$

$$= \frac{n(n+1)}{2n+1}\int_{-1}^{1}(P_{n+1})^2\,dx + \frac{n^2}{2n+1}\int_{-1}^{1}P_{n-1}\,\cancel{dx}\,P_{n+1}\,dx.$$

$$= \frac{n(n+1)}{2n+1}\left(\frac{2}{2(n+1)+1}\right) + 0.$$

$$\left(\because \int_{-1}^{1}P_n^2 = \frac{2}{2n+1}\right)$$

$$= \frac{2n(n+1)}{(2n+1)(2n+3)}$$

$$\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\, dx = \frac{2n(n+1)}{(2n+1)(2n+3)(2n-1)}$$

using recurrence I,

$$(2n+1)\, x P_n = (n+1) P_{n+1} + n \cdot P_{n-1} \quad \text{--- (1)}$$

Replace $n \to n+1$

$$(2n+3)\, x P_{n+1} = (n+2) P_{n+2} + (n+1) P_n \quad \text{--- (2)}$$

Replace $n \to n-1$

$$(2n-1)\, x P_{n-1} = n P_n + (n-1) P_{n-2} \quad \text{--- (3)}$$

Multiplying (2) & (3),

$$(2n-1)(2n+3)\, x^2 P_{n+1} . P_{n-1} = \left[(n+2)P_{n+2} + (n+1) P_n\right]\left[n P_n + (n-1) P_{n-2}\right]$$

$$(2n-1)(2n+3)\, x^2 P_{n+1} P_{n-1} = n(n+2) P_n P_{n+2} + (n+2)(n-1) P_{n-2} P_{n+2} + n(n+1) P_n^2 + (n^2-1) P_n P_{n-2}$$

$$\int_{-1}^{1} (2n-1)(2n+3)\, x^2 P_{n+1} P_{n-1} = \int_{-1}^{1} \cancelto{0}{n(n+2) P_n P_{n+2}}\, dx + \int_{-1}^{1} \cancelto{0}{(n+2)(n-1) P_{n-2} P_{n+2}}\, dx + \int_{-1}^{1} n(n+1) P_n^2\, dx + \int_{-1}^{1} \cancelto{0}{(n^2-1) P_n P_{n-2}}\, dx$$

Bessel's functions:

$x^2\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} + (x^2-n^2)y = 0$ is called Bessel's different-

tal ~~eq~~ equation, and the particular solutions are called Bessel's functions of order n.

$P_0(0) = 0$

$\therefore\ P_1(x) = (x-0)\cdot\frac{x}{x^2} = 1$ $\quad q_1(x) = \cancel{(x-0)^2}\cdot\frac{(x^2-n^2)}{\cancel{x^2}}$

$= -n^2.$

$\therefore$ '0' is a regular singular point.

$\therefore$ series solution is assumed as,

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots\ a_nx^{m+n}$

$\frac{dy}{dx} = a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots$

$\frac{d^2y}{dx^2} = a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+1)(m+2)x^m + \cdots$

coefficient of x^m,

$a_0[m^2-n^2] = 0.$

$a_0 \neq 0 \Rightarrow \boxed{m = \pm n.}$

Method-2:

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}.$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}.$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}.$$

Placing in Bessel's eq^n,

$$x^2\left[\sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}\right] + x\left[\sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}\right]$$

$$+ (x^2-n^2) \sum a_r x^{m+r} = 0.$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \left[(m+r)^2 - n^2\right] x^{m+r} + \sum a_r x^{m+r+2} = 0.$$

Equating coefficient of x^m $(r=0)$,

$$a_0[m^2-n^2] = 0.$$

$$a_0 \neq 0,$$

$$\boxed{m = \pm n}$$

equating coefficient of x^{m+1} $(r=1)$

$$a_1[(m+1)^2-n^2] = 0 \Rightarrow \boxed{a_1 = 0}$$

All odd coefficients are 0 ; $a_1 = a_3 = a_5 \cdots = 0$.

equating coefficient of x^{m+r+2} to zero,

$$a_{r+2}\left[(m+r+2)^2 - n^2\right] + a_r = 0$$

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$$a_{r+2} = \frac{-a_r}{(m+r+2)^2 - n^2}$$

for, $r=0$,

$$a_2 = \frac{-a_0}{(m+2)^2 - n^2}$$

$r=2$,

$$a_4 = \frac{-a_2}{(m+4)^2 - n^2} = \frac{a_0}{(m+2)^2 - n^2}\left(\frac{1}{(m+4)^2 - n^2}\right)$$

~~for~~ $m = \cancel{n}$,

$$\therefore y = a_0 x^m - \frac{a_0}{(m+2)^2 - n^2} x^{m+2} + \frac{a_0}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} + \cdots$$

for $m=n$,

$$y = \cancel{a_0 x^m} - \frac{a_0 x^2}{4(n+1)} +$$

$$y = a_0 x^m \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2 2!(n+1)(n+2)} \cdots\cdots\right]$$

for $m=-n$,

$$y = a_0 x^{-n}\left[1 - \frac{1 x^2}{4(-n+1)} + \frac{x^4}{4^2 2!(-n+1)(-n+2)} + \cdots\cdots\right]$$

$$a_0 = \frac{1}{2^n \gamma(n+1)}$$

$$\therefore J_n(x) = \frac{1}{2^n \gamma(n+1)} \frac{\Sigma (-1)^r x^{n+2r}}{2^{2r} r! (n+1)(n+2)\cdots\cdots(n+r)}$$

$\therefore$ for $m=n$,

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}}$$

where, $(n+r)! = \gamma(n+r+1)$

$\therefore$ for $m=-n$.

$$\boxed{J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\cancel{\Gamma n}\,\gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}}$$

~~$J_n(x)$~~

when n is not integral or zero,

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

General solution of Bessel's equation:

$$y = A J_n(x) + B J_{-n}(x)$$

$J_n(x) \rightarrow$ Bessel's function.

$$J_n(x) = a_0 x^n \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x^{2r}}{2^{2r}\, r!\,(n+1)(n+2) \cdots (n+r)}$$

Recurrence Relationships:

I. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k} \quad \text{--- (1)}$$

Multiplying both sides by ~~xn~~ x^n.
(of (1))

$$x^n . J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\gamma(n+k+1)\,k!} \frac{x^{n+2k+n}}{2^{n+2k}}.$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx}\left[\frac{\sum(-1)^k}{\gamma(n+k+1)\cdot k!} \cdot \frac{x^{2n+2k}}{2^{n+2k}}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k) \cdot x^{2n+2k-1}}{k!\,\gamma(n+k+1)\,2^{n+2k}}.$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k)}{k!\,\gamma(n+k+1)} \frac{x^{n+2k-1}}{2^{n+2k}}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k)}{k!\,\gamma(n+k)\,(n+k)} \frac{x^{n+2k-1}}{2^{n+2k}}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k\, 2\cancel{(n+k)}}{k!\,[\gamma(n+k)]\cancel{(n+k)}} \cdot \frac{x^{n+2k-1}}{2^{n+2k}}$$

$$= x^n \frac{(-1)^k}{k!\,\gamma(n+k)} \left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n \frac{(-1)^k}{k!\,\gamma(n-1+k+1)} \cdot \left(\frac{x}{2}\right)^{n-1+2k}.$$

$$\therefore \boxed{\frac{d}{dx}\left[x^n . J_n(x)\right] = x^n . J_{n-1}(x)}$$

II. $\frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \cdot \left(\frac{x}{2}\right)^{n+2k}.$$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \cdot \frac{x^{\cancel{n}+2k-\cancel{n}}}{2^{n+2k}}.$$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \frac{x^{2k}}{2^{n+2k}}.$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \cdot \frac{x^{2k}}{2^{n+2k}}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \cdot \frac{2k \cdot x^{2k-1}}{2^{n+2k}}.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(-1)\,k!\,\gamma(n+k+1)} \cdot \frac{2k\, x^{2k-1+n} \cdot x^{-n}}{2^{n+2k}}.$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 2\cancel{k}}{\cancel{k}(k-1)!\,\gamma(n+k+1)} \cdot \frac{x^{n+2k-1}}{2^{n+2k}}.$$

$$= -x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!\,\gamma(n+k+1)} \cdot \left(\frac{x}{2}\right)^{n+2k-1}$$

$k-1=r$

$k=r+1$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\cdot\gamma(n+1+r+1)} \left(\frac{x}{2}\right)^{n+2k-1}$$

$$-x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+1+r+1)} \cdot \left(\frac{x}{2}\right)^{n+1+2r}$$

$$\frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

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$$\boxed{\therefore J_0'(x) = -J_1(x)}$$

III. $J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$

$$\frac{d}{dx}\left[x^n \cdot J_n(x)\right] = x^n \cdot J_{n-1}(x)$$

$$n \cdot x^{n-1} \cdot J_n(x) + J_n'(x) \cdot x^n = x^n \cdot J_{n-1}(x).$$

$$\frac{n \cdot \cancel{x^n}}{x \cdot \cancel{x^n}} \cdot J_n(x) + J_n'(x) = J_{n-1}(x)$$

$$\boxed{J_n'(x) + \frac{n}{x} \cdot J_n(x) = J_{n-1}(x)}$$

IV. $J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$

From recurrence II,

$$\frac{d}{dx}\left[x^{-n} \cdot J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

$$-n \cdot x^{-n-1} \cdot J_n(x) + J_n'(x) \cdot x^{-n} = -x^{-n} J_{n+1}(x).$$

$$\frac{-n \cdot \cancel{x^{-n}}}{x} J_n(x) + J_n'(x) \cancel{x^{-n}} = -\cancel{x^{-n}} J_{n+1}(x)$$

$$\boxed{J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)}$$

V. $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

Adding recurrence III & IV,

$$J_n'(x) + \frac{n}{x} \cdot \cancel{J_n(x)} = J_{n-1}(x)$$

$$J_n'(x) - \frac{n}{x} \cancel{J_n(x)} = -J_{n+1}(x)$$

$$\boxed{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$$

VI. $\frac{2n}{x} \cdot J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Subtracting recurrences III & IV.

$$\cancel{J_n'(x)} + \frac{n}{x} J_n(x) = J_{n-1}(x)$$

$$\cancel{J_n'(x)} - \frac{n}{x} J_n(x) = -J_{n+1}(x)$$

$$\boxed{\frac{2n}{x} \cdot J_n(x) = J_{n-1}(x) + J_{n+1}(x)}$$

Q. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1)} \cdot \left(\frac{x}{2}\right)^{n+2k}.$$

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\gamma(n+k+1+1/2)} \cdot \left(\frac{x}{2}\right)^{2k+1/2}$$

$$= \frac{1}{\gamma(3/2)} \left(\frac{x}{2}\right)^{1/2} - \frac{1}{\gamma(5/2)} \cdot \left(\frac{x}{2}\right)^{5/2} + \frac{1}{2!} \frac{1}{\gamma(7/2)} \left(\frac{x}{2}\right)^{9/2} + \cdots$$

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$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\gamma(1/2)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\cdot\gamma(1/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\gamma(1/2)}\left(\frac{x}{2}\right)^4 + \cdots\right]$$

$$= \frac{\sqrt{x}\cdot 2}{\sqrt{2}\cdot\gamma(1/2)}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \frac{1}{x}\cdot\frac{\sqrt{x}\, 2}{\sqrt{2}\,\gamma(1/2)}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right].$$

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\sin x = \sqrt{\frac{2}{\pi x}}\sin x.$$

Q. $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot\cos x.$

$$J_{-n}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma(-\frac{1}{2}+r+1)}\left(\frac{x}{2}\right)^{-1/2+2r}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma(r+1/2)}\left(\frac{x}{2}\right)^{2r-1/2}$$

$$J_{-1/2}(x) = \frac{(x/2)^{-1/2}}{\gamma(1/2)} - \frac{(x/2)^{3/2}}{\gamma(3/2)} + \frac{\left(\frac{x}{2}\right)^{7/2}}{2!\,\gamma(5/2)} - \cdots \frac{(x/2)^{11/2}}{3!\,\gamma(9/2)}$$

$$= (x/2)^{+1/2}\left[\frac{(x/2)^{-1}}{\gamma(1/2)} - \frac{(x/2)^{1}}{\frac{1}{2}\gamma(1/2)} + \frac{(x/2)^{3}}{\frac{1}{2}\cdot\frac{3}{2}\cdot\gamma(1/2)} - \frac{(x/2)^{5}}{\frac{1}{2}\cdot\frac{3}{2}\cdot\gamma(1/2)} + \cdots\right]$$

$+\cdots$

$$= \frac{(x/2)^{1/2}}{\gamma(1/2)}\left[2x^{-1} - \frac{x/2}{1/2} + \frac{x^3}{\frac{2^3}{1/2\cdot 3/2}} - \frac{x^5/2^5}{1/2\cdot 3/2\cdot 5/2\cdot} - \cdots\right].$$

If difference of the roots is not

Revision

Q $x(1-x)\frac{d^2y}{dx^2} + 4\cdot\frac{dy}{dx} + 2y = 0$

$P_0(0) = 0.$

$Q_1(x) = (\cancel{x-0})\cdot\frac{4}{\cancel{x}(1-x)} = \frac{4}{1-x} = 4$

$Q_2(x) = (x-0)^2\cdot\frac{2}{x(1-x)} = = \frac{2x^2}{(1-x)\cancel{x}} = 0.$

∴ '0' is regular singular.

∴ $y = x^m a_0 + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \cdots a_n x^{m+n}.$

$\frac{dy}{dx} = m a_0 x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2}$

short-cut to find roots of the indicial equation:

$\cancel{r^2 + r(P_0 - 1) + q_0 = 0}$ $\quad$ $P_0 = \lim_{x\to 0} \frac{x P_1(x)}{P_0(x)} = Q_1(x)$

$r^2 + r(Q_1(x) - 1) + Q_2(x) = 0$ $\quad$ $q_0 = \lim_{x\to 0} \frac{x^2 P_2(x)}{P_0(x)} = Q_2(x).$

$\boxed{r^2 + r\left[\frac{4}{1-x} - 1\right] + \frac{2x}{1-x} = 0}$

$r^2 + r(4-1) + 0 = 0 \Rightarrow r^2 + 3r = 0 \Rightarrow r = 0, -3.$

Property of power series: all the terms are differentiable & continuous. This implies that the sum of all the terms must be continuous & differentiable.

$y = \sum_{n=0}^{\infty} c_n (x-x_0)^n$

Q $x^3 \cdot y'' + (\sin x) y = 0$

$$y'' + \frac{\sin x}{x^3} \cdot y = 0.$$

$$y'' + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}{x^3} y = 0.$$

$$y'' + \left(\frac{1}{x^2} - \frac{1}{3!} + \frac{x^2}{5!} - \dots\right) y = 0.$$

~~$Q_1(x) = x_0$~~

If $y_1(x)$ is L.I. solution of $y'' + P(x) \cdot y' + Q(x) \cdot y = 0$ then second L.I. solⁿ:

$$\boxed{y_2(x) = \int \left[\frac{e^{-\int P(x)dx}}{(y_1(x))^2}\right] dx.}$$

Legendre's series:

Q $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

~~$y = a_0 + a_1$~~

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots \quad a_n x^{m+n}$$

$$\frac{dy}{dx} = a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2}$$

$$+ a_4(m+4)x^{m+3} + \dots$$

$$\frac{d^2y}{dx^2} = a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$$

$$+ a_3(m+3)(m+2)x^{m+1} + a_4(m+3)(m+4)x^{m+2} + \dots$$

$\underline{\underline{m=0}}$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad a_n x^n.$$

$m=1$

$y = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots \quad a_n x^{n+1}$

coefficient of x^{m+r}

$$a_{r+2} = \frac{-(n+r+1)(n-r)}{(r+2)(r+1)} a_r.$$

Orthogonality of Bessel's function:

Parametric of Bessel's differential eq^n of order n:

$$x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (\alpha^2 x^2 - n^2) y = 0 \; ; \; n = 0, 1, 2 \dots$$

~~$y = C_1 J_n(x) + C_2 J$~~

$$y = C_1 J_n(\alpha x) + C_2 y_n(\alpha x)$$

$$\boxed{\int_0^1 x \cdot J_n(\alpha x) \cdot J_n(\beta x)\, dx = 0.}$$

where 'α' & 'β' are roots of the equation $J_n(x) = 0$

$$x^2 \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + (\alpha^2 x^2 - n^2) y = 0 \quad \text{——} \; (1)$$

$$x^2 \frac{d^2z}{dx^2} + x \cdot \frac{dz}{dx} + (\beta^2 x^2 - n^2) z = 0 \quad \text{——} \; (2)$$

solution of (1) & (2) are

$$y = J_n(\alpha x) \quad \& \quad z = J_n(\beta x)$$

Multiply (1) by z/x & (2) by $(-y/x)$ & add them.

$$z x \cdot \frac{d^2y}{dx^2} + z \cdot \frac{dy}{dx} + \frac{yz}{x} (\alpha^2 x^2 - n^2) = 0$$

$$- y x \cdot \frac{d^2z}{dx^2} - y \cdot \frac{dz}{dx} - \frac{yz}{x} (\beta^2 x^2 - n^2) = 0$$

$$x \left[z \frac{d^2y}{dx^2} - y \frac{d^2z}{dx^2} \right] + \left[z \frac{dy}{dx} - y \frac{dz}{dx} \right] + (\alpha^2 - \beta^2) x y z$$

$$\frac{d}{dx}\left[x\left(z\frac{dy}{dx}-y\cdot\frac{dz}{dx}\right)\right]+(\alpha^2-\beta^2)\,xyz=0 \quad \text{--- (3)}$$

Integrating (3) wrt 'x' & taking limits from '0' to '1'

$$\left[x\left(z\frac{dy}{dx}-y\cdot\frac{dz}{dx}\right)\right]_0^1+(\alpha^2-\beta^2)\int_0^1 xyz\,dx=0.$$

$$(\beta^2-\alpha^2)\int_0^1 xyz\,dx=\left[x\left(z\frac{dy}{dx}-y\cdot\frac{dz}{dx}\right)\right]_0^1$$

$$\cancel{(\beta^2-\alpha^2)} \quad = \left[z\frac{dy}{dx}-y\frac{dz}{dx}\right]_{x=0} \quad \text{--- (4)}$$

$y=J_n(\alpha x)$ & $z=J_n(\beta x)$

$\frac{dy}{dx}=\alpha J_n'(\alpha x)$; $\frac{dz}{dx}=\beta\cdot J_n'(\beta x)$

$$(\beta^2-\alpha^2)\int_0^1 x\cdot J_n(\alpha x)\cdot J_n(\beta x)\,dx=\Big[\alpha J_n'(\alpha x)\cdot J_n(\beta x)-\beta J_n'(\beta x)\cdot J_n(\alpha x)\Big]_{x=1}$$

$$=\alpha J_n'(\alpha)J_n(\beta)-\beta\cdot J_n'(\beta)J_n(\alpha)$$

'α' & 'β' are roots of $J_n(x)=0$; so $J_n(\alpha)=J_n(\beta)=0$.

$$\int_0^1 x\cdot J_n(\alpha x)\cdot J_n(\beta x)\,dx=0.$$

$$\int_0^1 x\left[J_n(\alpha x)^2\right]dx=\frac{1}{2}\left[J_n'(\alpha)\right]^2$$

$\therefore (\beta^2-\alpha^2)\int_0^1 x\cdot J_n(\alpha x)J_n(\beta x)\,dx=\alpha J_n'(\alpha)J_n(\beta)-\beta\cdot J_n'(\beta)J_n(\alpha)$

$\beta=\alpha$; $J_n(\alpha)=0$.

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$$\lim_{\beta\to\alpha} \int_0^1 x\cdot J_n(\alpha x)\cdot J_n(\beta x)\,dx = \lim_{\beta\to\alpha} \frac{\alpha J_n'(\alpha)\cdot J_n(\beta)}{(\beta^2-\alpha^2)}$$

$\left\{ \because \text{limit form is } \frac{0}{0} \right\}$

$$\int_0^1 x\left[J_n(\alpha x)\right]^2 dx = \lim_{\beta\to\alpha} \frac{\alpha J_n'(\alpha)\cdot J_n'(\beta)}{2\beta}$$

$$= \frac{1}{2}\left[J_n'(\alpha)\right]^2$$

generating function of $J_n(x)$:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = e^{\frac{xz}{2}}\cdot e^{-\frac{x}{2z}}$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

$$= J_0 + zJ_1 + z^2J_2 + z^3J_3 + \cdots$$
$$+ z^{-1}J_{-1} + z^{-2}J_{-2} + z^{-3}J_{-3} + \cdots$$

$\because J_{-n}(x) = (-1)^n J_n(x)$

$\therefore J_{-1}(x) = -J_1(x)$

$J_{-2}(x) = J_2(x)$

Trigonometric expansion involving bessel's function:

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + zJ_1 + z^2J_2 + \cdots + z^{-1}J_{-1} + z^{-2}J_{-2} + z^{-3}J_{-3} + \cdots$$

$z = e^{i\theta}$

$$e^{\frac{x}{2}\left(e^{i\theta}-\frac{1}{e^{i\theta}}\right)} = e^{\frac{x}{2}\left(e^{i\theta}-e^{-i\theta}\right)}$$ $\left\{ \because J_{-1}(x) = -J_1(x) \right\}$

$$= J_0 + zJ_1 + z^2J_2 + z^3J_3 + \cdots - J_1 z^{-1} + J_2 z^{-2} - J_3 z^{-3}$$

$$\Rightarrow e^{ix\sin\theta} = J_0 + J_1 e^{i\theta} + J_2 e^{2i\theta} - J_1 e^{-i\theta} + J_2 e^{-2i\theta} - J_3 e^{-3i\theta} \cdots$$

$$\Rightarrow \cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + J_1(e^{i\theta} - e^{-i\theta}) + J_2(e^{2i\theta} + e^{-2i\theta}) + J_3(e^{3i\theta} - e^{-3i\theta}) + \cdots$$

$$= J_0 + J_1(2i\sin\theta) + J_2(2\cos 2\theta) + \cdots$$

Equating real part & imaginary part,

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + \cdots$$

&

$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta$$

$$\theta = \pi/2 - \alpha$$

$\therefore$ $\cos x = J_0 - 2J_2 + 2J_4 + \cdots$

& $\sin x = 2J_1 - 2J_3 + 2J_5 \cdots$

Q $x^2\frac{d^2y}{dx^2} + xy' + (\alpha^2x^2 - n^2)y = 0.$

$$t = \alpha x.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \alpha\cdot\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \alpha^2\cdot\frac{d^2y}{dt^2}.$$

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Q. 1 $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$.

2. $\frac{d}{dx}\left\{x \cdot J_n(x) \cdot J_{n+1}(x)\right\} = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

3. $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

4. $(2x-1)^2\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2-2x+3$.

1. $J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$

$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma(n+r+1)}\left(\frac{x}{2}\right)^{5/2+2r}$

$J_{5/2}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma\left(\frac{7}{2}+r\right)}\left(\frac{x}{2}\right)^{5/2+2r}$

1. using recurrence VI,

$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

$\frac{\cancel{2}\cdot \frac{1}{\cancel{2}}}{x}J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x)$ $\left(n=\frac{1}{2}\right)$

$\frac{J_{1/2}(x)}{x} = J_{-1/2}(x) + J_{3/2}(x)$.

$\because J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot\sin x$

& $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$,

$$\therefore \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cdot\sin x = \sqrt{\frac{2}{\pi x}}\cos x + J_{3/2}(x)$$

$$\therefore J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cdot\sin x - \sqrt{\frac{2}{\pi x}}\cos x.$$

Again using VI, (placing $n=3/2$)

$$\frac{\cancel{2}\cdot 3/\cancel{2}}{x}\cdot J_{3/2}(x) = J_{1/2}(x) + J_{5/2}(x)$$

$$\frac{3}{x}\left[\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x\right] = \sqrt{\frac{2}{\pi x}}\sin x + J_{5/2}(x)$$

$$\therefore J_{5/2}(x) = \left(\frac{3\sin x}{x^2} - \frac{3}{x}\cos x - \sin x\right)\sqrt{\frac{2}{\pi x}}.$$

$$\boxed{J_{5/2}(x) = \left(\frac{3-x^2}{x^2}\sin x - \frac{3\cos x}{x}\right)\sqrt{\frac{2}{\pi x}}}$$

P. $$\frac{d}{dx}\left[x\cdot J_n(x)\cdot J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

$$J_n(x)\cdot J_{n+1}(x) + x\left[J_n'(x)\cdot J_{n+1}(x) + J_{n+1}'(x)\cdot J_n(x)\right]$$

using recurrence IV,

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x).$$

$$J_n(x)\cdot J_{n+1}(x) + x\left[\left(\frac{n}{x}J_n(x) - J_{n+1}(x)\right)J_{n+1}(x) + J_{n+1}'(x)\cdot J_n(x)\right]$$

$$J_n(x)\cdot J_{n+1}(x) + x\left[\frac{n}{x}J_n(x)J_{n+1}(x) - J_{n+1}^2(x) + J_n(x)\cdot J_{n+1}'(x)\right]$$

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Substituting ~~(n+1)~~ $n \to (n+1)$ in recurrence III,

$$J_{n+1}'(x) + \frac{n+1}{x} J_{n+1}(x) = J_{n+1-1}(x)$$

$$J_{n+1}'(x) = J_n(x) - \frac{n+1}{x} J_{n+1}(x)$$

$$\therefore J_n(x) J_{n+1}(x) + x \left[\frac{n}{x} \cdot J_n(x) J_{n+1}(x) - J_{n+1}^2(x) + J_n(x) \left(J_n(x) - \frac{(n+1)}{x} J_{n+1}(x) \right) \right]$$

$$J_n(x) J_{n+1}(x) + \frac{n J_n(x) J_{n+1}(x)}{x} x - x \cdot J_{n+1}^2(x) + x J_n(x) \left(J_n(x) - \frac{(n+1)}{x} J_{n+1}(x) \right)$$

$$\cancel{J_n(x) J_{n+1}(x)} + n J_n(x) J_{n+1}(x) - x J_{n+1}^2(x) + x J_n^2(x) - n J_{n+1}(x) - \cancel{J_n(x) \cdot J_{n+1}(x)}.$$

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