4/1/19

Syllabus

Module 1: Ordinary differential Equations – I, linear differential equations, Wronskian, linear independence and dependence of solutions, linear differential equations of 2nd order & higher order with constant coefficients; operator methods (simmons) legendre's & Euler – Cauchy's form of linear differential equation, method of variation of parameters.

Module 2: Ordinary differential equation – II, Ordinary & singular points of differential equation, Power & frobenius' series solution. Bessel's differential equation, Bessel function of first kind & its properties, legendre's differential equation, legendre's polynomial and its properties.

Module 3: (until Midsem) fourier series & partial differential equations fourier series: Euler's formula for fourier series, Dirichlet conditions, Half-range fourier series.

Textbooks:

1. E. Kreyszig, Advanced Engineering Mathematics (9th Edition)
2. D.G. Zill & W.S Wright, Advanced Engg. Maths. (4th Edition)
3. J.W. Brown & R.V. Churchill, complex variables & applications (7th
4. R.K. Jain & S.R.K. Iyengar, Advanced engineering Mathematics (3rd
5. R.A. Johnson & I. Miller & J. Freund : Probability & statistics for engineers
6. S.C. Gupta & V.K. Kapoor : fundamental of Mathematical statistics, Sultan chand & sons.

Reference Books:

1. G.F. Simmons, Differential Equations & its applications and Historical notes, ~~2nd~~ 2nd Edition.
2. N.P. Bali & Manish Goyal, A textbook of engineering mathematics, laxmi publications, Reprint, 2008.
3. E.A. Coddington, An introduction to ordinary differential

equations
4. W.E Boyce & R.C. DiPrima
5. P.L. Meyer.

Differential Equations

A differential equation is an equation containing derivative of an unknown function ~~variable~~, independent & dependent variables.

Ex: $\frac{dy}{dx} = e^x$; order = 1

Linear second order differential equations with constant coeff.

Let us consider a general linear equation of second order:

$$\boxed{\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + q \cdot y = v(x)} \; ; P, q : constant \quad (1)$$

The complete solution of this equation is

$$\boxed{y = u + w} \quad (2)$$

w: Any function which satisfies the given differential equation.

u: general solutions / complementary solutions

To find general solution, 'u', equate L.H.S. to 0. i.e.

$$\boxed{\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + q \cdot y = 0} \quad (3)$$

$y = u + w$

complementary solution — particular integral

complimentary function must be general solution of (3) and envolues 2 arbitrary constants. Arbitrary constants depend on order of differential equations.

A differential equation can be written as (3 Methods):

(1) $\frac{d^2y}{dx^2} + p \cdot \frac{dy}{dx} + q \cdot y = v(x)$

(2) $y'' + py' + qy = v(x)$

(3) $(D^2 + p \cdot D + q)\, y = v(x)$; $\left\{ D = \frac{d}{dx} \right\}$

$\therefore$ $Dy = \frac{dy}{dx}$ { 'D' is called the differential operator }.

Linear second order eq^n

Homogeneous

$\frac{d^2y}{dx^2} + p \cdot \frac{dy}{dx} + q \cdot y = 0$

Non-Homogeneous

$\frac{d^2y}{dx^2} + p \cdot \frac{dy}{dx} + q \cdot y = \cancel{v(x)}$ $v(x)$.

I. Finding general solution

consider a homogeneous equation,

$y'' + py' + qy = 0$ —— (A)

let $y = e^{mx}$; (m = constant)

$\frac{dy}{dx} = me^{mx}$ & $\frac{d^2y}{dx^2} = m^2 e^{mx}$.

~~as~~ exponential function ~~has the~~ e^{mx} has the property that its derivative are all constant multiples of the function itself.

Placing the values in eq^n (A),

$\therefore \; m^2 e^{mx} + p \cdot m \cdot e^{mx} + q \cdot e^{mx} = 0.$

$\Rightarrow e^{mx}(m^2 + pm + q) = 0$

$\because e^{mx} \neq 0,$

$\therefore \boxed{m^2 + pm + q = 0}$: auxiliory equation / quodratic equation

$$m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

~~whenever.~~

$\because D = \frac{d}{dx}$ & $Dy = \frac{dy}{dx}$,

replace ~~m=~~ $\boxed{D = m \text{ \& } D^2 = m^2}$

Let the roots of the equation be m_1 & m_2.

$\therefore$ the solution is $y_1 = e^{m_1 x}$. & $y_2 = e^{m_2 x}$.

CASE: 1:

→ when roots are real & distinct:

$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$; c_1 & c_2 are constants.

when $\frac{y_1}{y_2} \neq$ constant : linear independent

i.e. $\frac{e^{m_1 x}}{e^{m_2 x}} \neq$ constant. $= e^{(m_1 - m_2)x}$

Auxiliary equation is on ~~the form~~ terms of 'm'

Ex: converting equation into auxiliary equation:

say the given equation is :

$$\frac{d^2y}{dx^2} + 2 \cdot \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = Dy$$

$\therefore \; D^2y + 2Dy + y = 0 \Rightarrow (D^2 + 2D + 1)y = 0$

$\therefore$ Auxiliary equation is $\boxed{m^2+2m+1=0}$

(solutions are linearly independent)

linearly independent means not a constant multiple of the other. Functions defined on interval are not constant multiple of the other. or Wronskian of the solution is not equal to 0.

Q convert to auxiliary equations:

(i) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

$Dy = \frac{dy}{dx}$

$\therefore D^2y - 5Dy + 6y = 0$

$y(D^2 - 5D + 6) = 0$

$\therefore$ Auxiliary equation is $m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$(m-2)(m-3) = 0$

$\Rightarrow m = 2, 3.$

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\boxed{y = c_1 e^{2x} + c_2 e^{3x}}$

(ii) $\frac{d^2y}{dx^2} + y = 0$

$D^2y + y = 0$

$(D^2+1)y = 0$

$\therefore$ Auxiliary equation is: $m^2+1=0$

(iii) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 0$

$D^2y + 7Dy + 10y = 0$

$y(D^2 + 7D + 10) = 0$

$m^2 + 7m + 10 = 0.$

Wronskian:

Let ϕ_1 & ϕ_2 be the solutions of a differential equation.

∴ Wronskian is

$$\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} \neq 0 \quad : \text{linear independence.}$$

where $\phi_1 = e^{m_1 x}$

$\phi_2 = e^{m_2 x}$

Q $y'' - y' - 6y = 0.$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

$\cancel{m^2}$ $D^2y - Dy - 6y = 0$

$(D^2 - D - 6)y = 0$

∴ auxiliary equation is $m^2 - m - 6 = 0$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m+2)(m-3) = 0$

∴ $m = 3, -2$

$m_1 = 3$

$m_2 = -2$

∴ $y = c_1 e^{3x} + c_2 e^{-2x}.$

To check linear independence,

$$= \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -2e^{3x-2x} - 3e^{3x-2x}$$

$$= -5e^{x} \neq 0$$

∴ The solutions are linearly independent.

CASE 2: Distinct complex roots:

Let m_1 & m_2 be '$a+ib$' & '$a-ib$' respectively.

& ∵ $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\therefore e^{m_1 x} = e^{(a+ib)x} = e^{ax} \cdot e^{ibx} = e^{ax}(\cos bx + i\sin bx) \quad \text{—①}$$

$$\& \quad e^{m_2 x} = e^{(a-ib)x} = e^{ax} e^{-ibx} = e^{ax}(\cos bx - i\sin bx) \quad \text{—②}$$

{ ∵ $\cos(-\theta) = \cos\theta$ }

∵ we are interested in solutions that are real-valued,

∴ Adding ① & ② we get,

$$2e^{ax}\cos bx$$

Dividing by 2,

$$e^{ax}\cos bx.$$

Subtracting ① & ② & dividing by $2i$,

$$e^{ax}\sin bx$$

$$\therefore y = e^{ax}(c_1\cos bx + c_2\sin bx).$$

Q. $y'' + 2y' + 3y = 0$

$m^2 + 2m + 3 = 0$ (auxiliary equation)

$$m = \frac{-2 \pm \sqrt{4-12}}{2}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

$$= -1 \pm \sqrt{2}i$$

$\therefore m_1 = -1 + \sqrt{2}i$, $m_2 = -1 - \sqrt{2}i$; $a = -1$, $b = \sqrt{2}$.

$\therefore$ solution is $e^{ax}(c_1 \cos bx + c_2 \sin bx) = y$

$$\boxed{\therefore y = e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)}$$

CASE 3: when roots are equal & real.

'm_1' & 'm_2' are equal roots (real), with $m_1 = m_2 = -P/2$.

$\therefore$ our solution is assumed to be $y = e^{mx}$,

Thus,

$$y_1 = e^{-P/2\,x}.$$

$$v = \int \frac{e^{-\int P dx}}{y_1^2} = x$$

$$y_2 = vy_1 = xe^{mx}$$

$$\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$$

Q. $y'' + 8y' + 16y = 0.$

Auxiliary equation

$m^2 + 8m + 16 = 0$

$\therefore m = -4, -4.$

$\therefore y = c_1 e^{mx} + c_2 x e^{mx}$

$y = c_1 e^{-4x} + c_2 4 e^{-4x}$

Q. $y'' + y' - 6y = 0$

$m^2 + m - 6 = 0$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$\therefore m = 2, -3$

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-3x}}$$

II. finding particular solution (w)

$y'' + py' + qy = v(x)$ —— (1)

$\therefore$ complete solution of this differential equation = complementary function + particular integral

complementary function:

Distinct & real roots | equal roots | complex roots.

Equation (1) can be written as,

$P(D) \cdot y = v(x)$; $P(D) = D^2 + p \cdot D + q.$

$\therefore$ Particular integral $= \dfrac{v(x)}{P(D)}$; $P(D) = (D - r_1)(D - r_2)$

where r_1 & r_2 are roots of the auxiliary equation.

What does $\frac{1}{P(D)}$ represent?

If $Dy = f(x)$

$y = \frac{f(x)}{D}$

$y = \int f(x)\,dx$

'$\frac{1}{D}$' means integrating the function once.

If 'D' is in the numerator, differentiate the function once.

$(D-r)\,y = f(x)$; r = constant

$y = \frac{f(x)}{(D-r)}$

$\left(\frac{d}{dx} - r\right) y = f(x)$

$\frac{dy}{dx} - ry = f(x)$ {first order linear differential eq^n: $\frac{dy}{dx} + P\cdot y = Q$ }

Integrating factor $= e^{\int -r\,dx} = e^{-rx}$ ($\because r$ is a constant)

$\therefore$ solution is

$e^{-rx}\cdot y = \int e^{-rx}\cdot f(x)\,dx$

(y) $y = e^{rx}\int e^{-rx} f(x)\,dx$

$\frac{y(x)}{(D-r_1)(D-r_2)}$ Apply $(D-r_1)$ on $y(x)$ using the above to obtain (y). On this 'y' apply $(D-r_2)$ again using the same formula.

Q. $y'' + 4y' + 4y = 10x^3 e^{-2x}$

$$P.I. = y = \frac{V(x)}{P(D)}$$

$$P.I. = \frac{10x^3 e^{-2x}}{(D+2)(D+2)}$$

$$y = e^{rx} \int e^{-rx} \cdot f(x)\, dx$$

$$= e^{-2x} \int \cancel{e^{2x}} (10x^3 \cancel{e^{-2x}})\, dx$$

$$= e^{-2x} \cdot \frac{10x^4}{4}$$

$$y = \frac{5}{2} x^4 \cdot e^{-2x}$$

Now apply ~~D+2~~ $(D-r_2)$ ie. $(D+2)$ on this 'y'

$$\therefore y = \frac{\cancel{10}\ \frac{5x^4 e^{-2x}}{2}}{(D+2)}$$

Here, again $r = -2$

$$\therefore y = e^{rx} \int e^{-rx} f(x)\, dx$$; Now $f(x)$ is $5/2\, x^4 \cdot e^{-2x}$

$$= e^{-2x} \int \cancel{e^{2x}} \cdot \frac{5}{2} x^4 \cdot \cancel{e^{-2x}}\, dx$$

$$= \frac{\cancel{5}}{2} e^{-2x} \cdot \frac{x^5}{\cancel{5}}$$

$$\boxed{y = \frac{x^5}{2} \cdot e^{-2x}}$$

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Cases for Particular integral:

case 1: $v(x) = e^{ax}$

case 2: $v(x) = \cos(ax+b)$ or $\sin(ax+b)$

case 3: $v(x) = x^m$

case 4: $v(x) = e^{ax} \cdot \cancel{v(x)}\, V$. $V(x)$

case 5: $\cancel{x}$ is another function of x.

CASE 1: $v(x) = e^{ax}$

$$y'' + p \cdot y' + q \cdot y = e^{ax}$$

$$P(D) \cdot y = e^{ax}$$

$$y = \frac{e^{ax}}{\cancel{D^2} * P(D)} = \frac{e^{ax}}{D^2 + p \cdot D + q}$$

when $\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$; $f(a) \neq 0$

Replace, 'D' by 'a'.

$$\therefore \quad P.I. = \frac{e^{ax}}{D^2 + p \cdot D + q}$$

$$P.I = \frac{e^{ax}}{a^2 + p \cdot a + q}$$

Q $(D^2 - 2D + 1)\, y = 3e^{5/2 x}$.

$$P.I. = \frac{v(x)}{P(D)} = \frac{3e^{5/2 x}}{D^2 - 2D + 1}$$

$$= \frac{3e^{5/2 x}}{a^2 - 2a + 1} \quad ; a = 5/2$$

$$= \frac{3e^{5/2 x}}{\frac{25}{4} - \cancel{2} \cdot \frac{5}{\cancel{2}} + 1} = \frac{3e^{5/2 x}}{25 + 4 - 20} \times 4 = \frac{12\, e^{5/2 x}}{9}$$

$$= \frac{4}{3} \cdot e^{5/2 x}.$$

$$\boxed{\therefore P.I. = 4/3\, e^{5/2 x}}$$

C.F.

Auxiliary solution = m^2-2m+1

$(m-1)^2=0$

$m=1,1$

$y = C.F. = c_1e^{x} + c_2 \cdot xe^{x}.$

2. $\frac{d^2y}{dx^2} + y = e^{-x}$ $= D^2y+y = e^{-x}$

$(D^2+1)y = e^{-x}$

$P.I. = \frac{V(x)}{P(D)} = \frac{e^{-x}}{2}$

C.F.

Auxiliary equation:

$m^2+1=0$

$m^2=-1$

$m = \pm\sqrt{-1}$

$m = \pm i$

$C.F. = e^{ax}(c_1\cos bx + c_2\sin bx)$ $a=0$

$= e^{0}(c_1\cos x + c_2\sin x)$ $b=1.$

$= c_1\cos x + c_2\sin x.$

$\therefore$ complete solution : complementary function + Particular Integral

$= c_1\cos x + c_2\sin x + \frac{e^{-x}}{2}$

When $f(a)=0$:

Ex: $(D^2-2D+1)\,y = e^{x}$

$\frac{d^2y}{dx^2} + 3\cdot\frac{dy}{dx} + 2y = e^{-x}$

$(D^2+3D+2)\,y = e^{-x}$

$P.I. = \frac{V(x)}{P(D)}$

$= \frac{e^{-x}}{D^2+3D+2}$; $a=-1$

$= \frac{e^{-x}}{a^2+3a+2}$

$= \frac{e^{-x}}{0}$

$\therefore$ finding $f'(D) = 2D+3$.

$f'(a) = f'(-1) = 1$

$\therefore P.I. = \frac{x\cdot e^{-x}}{f'(a)} = \frac{x\cdot e^{-x}}{1}$ $\cancel{\cdot \frac{x^1}{1!}\cdot\frac{1}{2}\, e^{-x}} = \cancel{\frac{xe^{-x}}{2}}$ $\frac{xe^{-x}}{1} = xe^{-x}$

For $f(a)=0$, $f^{r-1}(a)=0$

but $f^{r}(a)\neq 0$.

$\boxed{P.I. = \frac{x^r}{r!}\cdot\frac{1}{f^r(a)}\cdot e^{ax}}$

CASE 2: $\cos nx$ or $\sin nx$

$P(D).y = \cos nx$

$y = \frac{\cos nx}{P(D)} = \frac{\cos nx}{D^2+p.D+q}$ $= \cancel{\cos x}$

Replace $D^2 = -n^2$

$\frac{\cos nx}{-n^2+p\cdot D+q} = \frac{\cos nx}{p\cdot D+(q-n^2)}$

Rationalising,

$\frac{\cos nx}{p D+q-n^2} \times \frac{p\cdot D-(q-n^2)}{p\cdot D-(q-n^2)} = \frac{\cos nx\,(pD-(q-n^2))}{p^2D^2-(q-n^2)^2}$

$= \frac{p\cdot D(\cos nx)-\cos nx(q-n^2)}{p^2D^2-(q-n^2)^2}$

$= \frac{-pn\sin nx-\cos nx(q-n^2)}{p^2D^2-(q-n^2)^2}$

Q. $\frac{d^2y}{dx^2}+y=12\sin 2x$.

$(D^2+1)y=12\sin 2x$.

$m^2+1=0$

$m=\pm i=0\pm i\cdot 1$

$\therefore CF=e^{ax}(c_1\cos bx+c_2\sin x)$

$a=0$

$b=1$

$CF=c_1\cos x+c_2\sin x$

$P.I.=y=\frac{12\sin 2x}{D^2+1}$

$D^2=-n^2$

$=-4$

$P.I.=y=\frac{12\sin 2x}{-4+1} = -4\sin 2x$.

$C.S.= -4\sin 2x+c_1\cos x+c_2\sin x$

$\frac{d^2y}{dx^2}-4\cdot\frac{dy}{dx}-5y=\sin x$

$(D^2-4D-5)y=\sin x$; $n=1$

$P.I.=\frac{\sin x}{D^2-4D-5}$

$\therefore P.I.=\frac{\sin x}{-1-5-4D} = \frac{-\sin x}{(4D+6)}$

$= \frac{-\sin x\,(4D-6)}{(4D+6)(4D-6)}$

$D^2=-n^2$

$=-1$

$$-\frac{\sin x(4D-6)}{16D^2-36}$$

$\because D^2=-1,$

$$\therefore \text{P.I.} = \cancel{-si}\ \frac{-4D(\sin x)+6\sin x}{-16-36}$$

$$= \frac{6\sin x-4\cdot\cos x}{-52}$$

$$= \frac{3\sin x-2\cos x}{-26}$$

$$\text{P.I.} = \frac{2\cos x-3\sin x}{26}.$$

Q. $\frac{d^2y}{dx^2}+2\cdot\frac{dy}{dx}+3y=\sin x$

$(D^2+2D+3)y=\sin x$

$$\text{P.I.} = \frac{\sin x}{D^2+2D+3}.$$

$D^2=-n^2$

$D^2=-1$

$$\text{P.I.} = \frac{\sin x}{-1+3+2D}$$

$$= \frac{\sin x}{2D+2}\cdot\frac{(2D-2)}{(2D-2)}$$

$$= \frac{2D\sin x-2\sin x}{4D^2-4}$$

$$= \frac{2\cos x-2\sin x}{-8}$$

$$= \frac{\sin x-\cos x}{4}$$

C.F.: $m^2+2m+3=0$

$$m=\frac{-2\pm\sqrt{4-12}}{2}$$

$= -1\pm\sqrt{2}i$

$a=-1$

$b=\sqrt{2}$

$\therefore CF = e^{ax}(c_1\cos bx+c_2\sin bx)$

$= e^{-x}(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x)$

$\therefore$ complete solution

$$\frac{\sin x-\cos x}{4}+e^{-x}(c_1\cos\sqrt{2}x+\sin\sqrt{2}x)$$

2. $y'' + 4y' + 4y = 3\sin x + 4\cos x$

$$(D^2+4D+4)y = 3\sin x + 4\cos x$$

$$y = \frac{3\sin x}{D^2+4D+4} + \frac{4\cos x}{D^2+4D+4}$$

$$D^2 = -1 \qquad D^2 = -1$$

$$= \frac{3\sin x}{-1+4+4D} + \frac{4\cos x}{-1+4D+4}$$

$$= \frac{3\sin x}{4D+3} + \frac{4\cos x}{4D+3}.$$

$$= \frac{3\sin x\,(4D-3)}{(4D+3)(4D-3)} + \frac{4\cos x\,(4D-3)}{(4D+3)(4D-3)}$$

$$= \frac{12D\sin x - 9\sin x}{16D^2-9} + \frac{16D\cos x - 12\cos x}{16D^2-9}$$

$$= \frac{12\cancel{\cos x} - 9\sin x}{-25} + \frac{-16\sin x - 12\cancel{\cos x}}{-25}$$

$$P.I. = \frac{-\cancel{25}\sin x}{-\cancel{25}} = \sin x.$$

~~∴ complete solution = C+~~

CF: $m^2+4m+4=0$

$$(m+2)^2 = 0$$

$$m = -2, -2$$

$\therefore$ $CF = c_1e^{-2x} + c_2xe^{-2x}$

$\therefore$ $CS = CF + PI$

$$= c_1e^{-2x} + c_2xe^{-2x} + \sin x.$$

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Q $(D^2+1)y = \sin x$

$P.I. = \frac{\sin x}{D^2+1}$

$D^2 = -n^2$

$D^2 = -1$

$\therefore P.I. = \frac{\sin x}{0}$

$f'(D) = 2D$.

$x \cdot \sin x$

$P.I. = \frac{\cancel{x^r}}{r!} \cdot \frac{\sin x \, f'(D)}{f^r(a)}$

$= \frac{x!}{1!} \cdot \frac{\sin x}{\cancel{f}'(a)}$

$= x \cdot \frac{\sin x \cdot D}{2D^2}$

$= \frac{x \cdot \cos x}{-2}$.

CF : $m^2+1=0$

$m = \pm i$

So $CF = c_1 \cos x + c_2 \sin x$

$CS = c_1 \cos x + c_2 \sin x - \frac{x \cos x}{2}$

Q.1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}$

$(D^2+3D+2)y = e^{-x}$; $a = -1$.

complementary function:

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0$

$(m+1)(m+2)=0$

$m = -1, -2$

$\boxed{\therefore y = c_1 e^{-x} + c_2 e^{-2x} = CF}$

Particular integral:

$P.I. = \frac{V(x)}{P(D)}$

$= \frac{e^{-x}}{D^2+3D+2}$

$= \frac{e^{-x}}{a^2+3a+2}$

$= \frac{e^{-x}}{1-3+2} = \frac{e^{-x}}{0}$.

$\therefore f(a) = 0$

$\therefore f'(D) = 2D+3$.

$f'(a) = f'(-1) = 1$.

$\therefore$ complete solution:

$\therefore$ P.I. $= \frac{x^r}{r!} \frac{V(x)}{f^r(a)}$

$y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x}$

$e^{-x}(c_1 + c_2 e^{-x} + x) = 0$

P.I. $= \frac{x^1}{1!} \frac{e^{-x}}{f'(-1)} = \frac{x \cdot e^{-x}}{1}$

$c_1 + c_2 e^{-x} + x = 0$.

$\frac{dy}{dx} = -x \cdot c_1 e^{-x} - 2c_2 e^{-2x} - x^2 e^{-x}$

$-2c_2 = 0$

$c_2 = 0$.

$c_1 + x = 0$

$c_1 = -x$

$\therefore y = xe^{-x}$

$y = c_1 e^{-2x} + c_2 e^{-x} + xe^{-x}$

$y = c_1 \quad 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$c_2 = -1$

$\frac{dy}{dx} = -x \cdot e^{-x} - 2x c_2 e^{-2x} + e^{-x} + -e^{-x} \cdot x = 0$

$-1 - 2c_2 + 1 - 0 = 0$.

$-1 - 2c_2 = 0$

$2c_2 = -1$.

$+2c_1 = +1$

$c_1 = 1/2$

$-2c_1 e^{-2x} - c_2 e^{-x} + e^{-x}$

$- xe^{-x}$

$-2c_1 - c_2 + 1 = 0$

$-c_1 = -1$

$c_1 = 1$

$e^{-x} \cdot e^{-2x} - e^{x} e^{-x}$

$S =$

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

$(D^2+D+1)\cdot y = \sin 2x$

$n = 2$

$D^2 = -n^2 = -4.$

$\therefore$ Particular integral $= \frac{V(x)}{P(D)}$

$= \frac{\sin 2x}{D^2+D+1}$

$= \frac{\sin 2x}{-4+1+D}$

$= \frac{\sin 2x}{D-3} \cdot \frac{(D+3)}{(D+3)}$

$= \frac{D(\sin 2x) + 3\cdot \sin 2x}{D^2-9}$

$= \frac{2\cos 2x + 3\cdot \sin 2x}{-13}$

$P.I. = -\frac{(2\cos 2x + 3\sin 2x)}{13}$

Complementary function:

$m^2+m+1=0$

$m = \frac{-1 \pm \sqrt{1-4}}{2}$

$m = \frac{-1 \pm \sqrt{3}i}{2} \quad = \quad a = -1/2$

$b = \frac{\sqrt{3}i}{2}$

$y = C.F. =$

$e^{ax}(\cos bx\, c_1 + c_2 \sin bx)$

$= e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

$\therefore C.S. = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}x}{2} + c_2 \sin \frac{\sqrt{3}x}{2}\right) + -\frac{(2\cos 2x + 3\sin 2x)}{13}$

3. $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 50y = \sin\omega x$

$(D^2+10D+50)y = \sin\omega x$

Particular integral:

$$P.I. = \frac{V(x)}{P(D)}$$

$$= \frac{\sin\omega x}{D^2+10D+50}$$

$D^2 = -n^2$

$= -\omega^2$

$$\therefore P.I. = \frac{\sin\omega x}{-\omega^2+10D+50}$$

$$= \frac{\sin\omega x}{50-\omega^2+10D} \cdot \frac{50-\omega^2-10D}{50-\omega^2-10D}$$

$$= \frac{(50-\omega^2)\sin\omega x - 10D(\sin\omega x)}{(50-\omega^2)^2 - 100D^2}$$

$$P.I. = \frac{(50-\omega^2)\sin\omega x - 10\omega\cos\omega x}{(50-\omega^2)^2 + 100\omega^2}$$

$$= \frac{(50-\omega^2)\sin\omega x - 10\omega\cos\omega x}{2500+\omega^4-\cancel{100\omega^2}+\cancel{100\omega^2}}$$

$$= \frac{(50-\omega^2)\sin\omega x - 10\omega\cos\omega x}{2500+\omega^4}.$$

Complementary function:

$$m^2+10m+50=0$$

$$m = \frac{-10 \pm \sqrt{100-200}}{2}$$

$$= \frac{-10 \pm \sqrt{-100}}{2} = \frac{-10 \pm 10i}{2} = -5 \pm 5i.$$

$\therefore CF = e^{ax}(c_1\cos bx + c_2\sin bx)$

$= e^{-5x}(c_1\cos 5x + c_2\sin 5x)$

$\therefore C.S. = \dfrac{(50-\omega^2)\sin\omega x + -10\omega\cos\omega x}{2500+\omega^4} + e^{-5x}(c_1\cos 5x + c_2\sin 5x)$

4. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = (1+\sin x)^2$

$\cos 2x = 1-2\sin^2 x$

$2\sin^2 x = 1-\cos 2x$

$(D^2+D+1)y = 1+\sin^2 x + 2\sin x = 1 + \dfrac{1-\cos 2x}{2} + 2\sin x$

$= \cancel{\dfrac{2-\cos 2x + 2\sin}{2}}$

$P.I. = \dfrac{(1+\sin x)^2}{D^2+D+1}$ $= \dfrac{3-\cos 2x + 2\sin x}{2}$

$\dfrac{1}{1-r} = 1+r^{0}+r^2+r^3+\dots = (1-r)^{-1}$

$\dfrac{1}{1+r} = 1-r+r^2-r^3+\dots = (1+r)^{-1}$

$P.I. = \dfrac{\dfrac{3-\cos 2x + 2\sin x}{2}}{D^2+D+1}$

$= \dfrac{3}{2(D^2+D+1)} \quad \dfrac{-\cos 2x}{D^2+D+1} \quad \dfrac{+2\sin x}{D^2+D+1}$

$= \dfrac{3\,\cancel{x}\,e^{0}}{2(1+(D+D^2))} \quad \dfrac{-\cos 2x}{D^2+D+1} \quad \dfrac{+2\sin x}{D^2+D+1}$

$=$

$= \dfrac{3}{2}(1+(D+D^2))^{-1} \quad \dfrac{-\cos 2x}{D^2+D+1} \quad \dfrac{+2\sin x}{D^2+D+1}.$

$= \dfrac{3}{2}\left(1-(D+D^2)+(D+D^2)^2-(D+D^2)^3+\dots\right) \quad \dfrac{-\cos 2x}{-4+1+D} \quad \dfrac{+2\sin x}{D.}$

$\dfrac{3}{2}\dfrac{e^{0}}{(0^2+0+1)} \quad - \dfrac{\cos 2x}{D-3} \quad \dfrac{+2\sin x.D}{D^2}$

$\frac{3}{2} - \frac{\cos 2x\,(D+3)}{(D^2-9)} + \frac{2\cos x}{-1}.$

$P.I. = \frac{3}{2} + \frac{2\sin 2x - 3\cos 2x}{-13} - 2\cos x.$

C.F.: $m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{-3}}{2}$

$m = \frac{-1 \pm \sqrt{3}i}{2}$

$y = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2 \cancel{\cos} \sin\frac{\sqrt{3}}{2}x\right).$

$C.S = \frac{3}{2} + \frac{3\cos 2x - 2\sin 2x}{13} - 2\cos x + e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx^2} - 2y = e^x + \cos x$

$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$

CF:

$m^3 - 3m^2 + 4m - 2 = 0$

$m = 1$

$$
\begin{array}{r|l}
m-1 & \cancel{m^3} - 3m^2 + 4m - 2 \,(\, m^2 - 2m + 2 \\
 & \cancel{m^3} - m^2 \\
 & - \quad + \\
\hline
 & -\cancel{2m^2} + 4m - 2 \\
 & -\cancel{2m^2} + 2m \\
 & + \quad - \\
\hline
 & 2m - 2 \\
 & \cancel{2m} - 2 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

$m^2 - 2m + 2 = 0$

$\cancel{m^2}$ $m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$m = 1, 1+i, 1-i$

$CF = e^{x}(c_1 \cos x + c_2 \sin x) + c_3 e^{1x}$

$$P.I = \frac{e^x + \cos x}{\cancel{D^3+} D^3 - 3D^2 + 4D - 2}$$

$$= \frac{e^x}{D^3 - 3D^2 + 4D - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{e^x}{1 - 3 + 4 - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2}$$

$$= \frac{e^x}{0}$$

$\therefore f'(D) = 3D^2 - 6D + 4$

$\cdot f'(1) = 3 + 4 - 6 = 1.$

CASE 3: $V(x) = e^{\alpha x} \sin nx$

$$P.I. = \frac{e^{\alpha x} \sin nx}{f(D)}$$

$$= e^{\alpha x} \cdot \frac{\sin nx}{f(D+\alpha)}$$

Exponential shift rule.

Q. $(D^4 - 1)y = e^x \cos x$

C.F.

$m^4 - 1 = 0$

$(m^2+1)(m^2-1) = 0$

$m = \pm 1, \pm i.$

$$\boxed{y = c_1 e^x + c_2 e^{-x} + e^{0x}(\cos x \cdot c_3 + c_4 \sin x)}$$

$P.I. = \dfrac{e^x \cos x}{D^4-1}$; $a=1$

$= e^x \cdot \dfrac{\cos x}{(D+1)^4-1^4}$

$= e^x \cdot \dfrac{\cos x}{[(D+1)^2-1][(D+1)^2+1]}$

$e^x \cdot \dfrac{\cos x}{(D^2+\cancel{1}+2D-\cancel{1})(D^2+2D+2)}$

$e^x \cdot \dfrac{\cos x}{(D^2+2D)(D^2+2D+2)}$

$D^2=-n^2 \Rightarrow D^2=-1$

$\therefore P.I. = \dfrac{e^x \cos x}{(2D-1)(2D+1)}$

$= e^x \cdot \dfrac{\cos x}{4D^2-1} = \dfrac{e^x \cos x}{-5}$

$$\boxed{P.I. \quad -\frac{e^x \cos x}{5}}$$

① $\dfrac{d^2y}{dx^2} + y = e^x \cdot \sin x$

$(D^2+1)y = e^x \sin x$

C.F:

$m^2+1=0$ $\quad$ $C.F = e^{0x}(c_1\cos x + c_2 \sin x)$

$m = \pm i$

$$P.I. = \frac{e^{x} \cdot \sin x}{D^2+1} \quad ; \alpha = 1$$

$$= e^{x} \cdot \frac{\sin x}{(D+1)^2+1}$$

$$= e^{x} \cdot \frac{\sin x}{D^2+1+2D+1}$$

$$= e^{x} \cdot \frac{\sin x}{2D+1}$$

$$= e^{x} \cdot \frac{\sin x}{(2D+1)} \frac{(2D-1)}{(2D-1)}$$

$$= e^{x} \cdot \frac{\sin x\,(2D-1)}{4D^2-1}$$

$$= e^{x} \cdot \frac{2D(\sin x) - \sin x}{-5}$$

$$= e^{x} \frac{(2\cos x - \sin x)}{-5}$$

$$P.I. = \frac{-e^{x}}{5} (2\cos x - \sin x)$$

CASE 4: $N(x) = x^{n} \cos mx$

$$P.I. = \frac{x^n \cdot \cos mx}{P(D)}$$

$$= x^n \cdot \frac{\text{Real part of } e^{imx}}{P(D)}$$

$$P.I. = R.P \text{ of } e^{imx} \cdot \frac{x^n}{P(D+im)}$$

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Q. $(D^4-1)y = x \cdot \sin x$

$$P.I. = \frac{x \cdot \sin x}{(D^4-1)}$$

$$= \frac{x \cdot (I.P \text{ of } e^{ix})}{(D^4-1)}$$

$$= (I.P. \text{ of } e^{ix}) \cdot \frac{x}{P(D+i)}$$

$$= (I.P. \text{ of } e^{ix}) \cdot \frac{x}{(D+i)^4 - 1^4}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{[(D+i)^2 - 1^2][(D+i)^2 + 1^2]}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{(D^2 - 1 + 2Di - 1)(D^2 \cancel{-1} + 2Di \cancel{+1})}$$

$D^2 = -1$ $(-m^2)$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{(2Di-3)(2Di-1)}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{4D^2(-1) - 2Di - 6Di + 3}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{4+3-8Di}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{x}{(7-8Di)} \frac{(7+8Di)}{(7+8Di)}$$

$$(I.P. \text{ of } e^{ix}) \cdot \frac{(7x + 8 \cdot i)}{49 - 64D^2 i^2}$$

$$(I.P. \text{ of } e^{ix}) \frac{(7x+8i)}{(49-64)} = \frac{(I.P. \text{ of } e^{ix})(7x+8i)}{-15}$$

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Q. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cdot \cos x$

Q. $\frac{d^4y}{dx^4} - y = x \cdot \cos x$.

Q. $(D^2 - 2D + 1) y = x \cdot e^x \sin x$

First applying exponential shift rule:

$$P.I. = e^x \cdot \frac{x \sin x}{D^2 - 2D + 1}$$

$$= e^x \cdot \frac{x \sin x}{(D-1)^2}$$

$$= e^x \cdot \frac{x \sin x}{D^2}.$$

$$e^x \cdot (I.P. \text{ of } e^{ix}) \cdot \frac{x}{D^2}$$

$$e^x \cdot (I.P. \text{ of } e^{ix}) \cdot \frac{x}{(D+i)^2}$$

$$e^x (I.P. \text{ of } e^{ix}) \cdot \frac{x}{(D^2 - 1 + 2Di)}$$

$$= e^x (I.P. \text{ of } e^{ix}) \frac{-x}{-[1 - (D^2 + 2Di)]}$$

$$= -e^x (I.P. \text{ of } e^{ix}) \cdot x (1 - (D^2 + 2Di))^{-1}$$

$$= -e^x (I.P. \text{ of } e^{ix}) \; x (1 + D^2 + 2Di + (D^2 + 2Di)^2 + \cdots)$$

$$= -e^x (I.P. \text{ of } e^{ix}) (x + 2i)$$

I.P. of $\left[e^{-x} \cdot (\cos x + i \sin x)(x + 2i) \right]$

I.P. of $\left\{ e^{-x} (x\cos x + x i \sin x + 2i \cos x - 2 \sin x) \right\} = (xe^{-x} \sin x + 2e^{-x} \cos x)$

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Q. $\frac{d^2y}{dx^2} - y = x \cdot e^x \sin x$

P.I. = $\frac{x \cdot e^x \sin x}{(D+1)^2 - 1}$. $(\alpha = 1)$

$= \cancel{x e^x \sin x}$ $e^x \cdot \frac{x \sin x}{D^2 + \cancel{1} + 2D - \cancel{1}}$

$e^x \cdot \frac{x \sin x}{D^2 + 2D}$ $= e^x \cdot \frac{x \cdot \text{I.P. of } e^{ix}}{(D+i)^2 + 2(D+i)}$

$e^x \frac{x \sin x}{D^2 + 2D - 1 + 1}$ $\frac{e^x \cdot (\text{I.P. of } e^{ix}) \cdot x}{D^2 - 1 + 2Di + 2D + 2i}$

$e^x \cdot \frac{x \sin x}{1 - (1 - 2D - D^2)}$

$= e^x \cdot \frac{(\text{I.P. of } e^{ix}) \cdot x}{D^2 + 2Di + 2D + 2i - 1}$

$e^x (\text{I.P. of } e^{ix}) \cdot \frac{x}{(2i-1)\left(1 + \frac{D^2 + 2Di + 2D}{2i-1}\right)}$

$e^x \cdot (\text{I.P. of } e^{ix}) \cdot (2i-1) \cdot x \left(1 + \frac{D^2 + 2Di + 2D}{2i-1}\right)^{-1}$

$e^x (\text{I.P. of } e^{ix}) (2i-1)\, x \left(1 - \frac{D^2 + 2Di + 2D}{2i-1} + \left(\frac{D^2 + 2Di + 2D}{2i1}\right)^2 \dots\right)$

$e^x (\text{I.P. of } e^{ix}) (2i-1) \left(x - \frac{1}{2i-1}(0 + 2i + 2)\right)$

$e^x (\text{I.P. of } e^{ix}) (2i-1) \left(x - \frac{2i+2}{2i-1}\right)$

$e^x (\text{I.P. of } e^{ix}) (2i-1) \left(x - \frac{2(i+1)(2i+1)}{(2i-1)(2i+1)}\right)$

 psx)

e^x (I.P. of e^{ix}) $(2i-1)\left(x - \frac{2(-2+3i+1)}{4i^2-1}\right)$

e^x(I.P. of e^{ix}) $(2i-1)\left(x-\frac{2(3i-1)}{-5}\right)$

$e^x\left(e^x \cdot \text{I.P.}\left((e^{ix})(2i-1)\left(x-\frac{(6i-2)}{-5}\right)\right)\right.$

$e^x \cdot \text{I.P.}\left[(\cos x+i\sin x)(2i-1)(x+6i-2)\right]$

$e^x \cdot \text{I.P.}\left[(\cos x+i\sin x)(2ix-12-4i-x-6i+2)\right]$

$e^x \cdot \text{I.P.}\left[(\cos x+i\sin x)(2ix-x-10-10i)\right]$

Cauchy's Euler's form of differential ~~dt~~ equation:

Dr

The differential equation in which the degree of the monomial & order of the equation is same.

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0.$$

* monomial → degree = 2
* derivative → order = 2.

let $y=x^m$

$$ax^2 \cdot m(m-1)\cdot x^{m-2}+bm\cdot x^m+cx^m=0$$

$$x^m\left(am^2+(b-a)m+c\right)=0$$

$$x^m \neq 0$$

case I: Distinct real roots

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

case II: Roots are equal

$$y = c_1 x^{m_1} + c_2 x^{m_2} \log x$$

case III. complex roots

$$y = x^{\alpha}\left[c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\right]$$

roots: $\alpha + i\beta$, $\alpha - i\beta$.

Q.1. $x^2 \cdot \frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} - 4y = 0.$

$$y = x^m.$$

$$x^m\left[1 \cdot m^2 + (-2-1)m - 4\right] = 0$$

$$x^m\left[m^2 - 3m - 4\right] = 0$$

$$m^2 - 4m + 3m - 4 = 0$$

$$m(m-4) + 1(m-4) = 0$$

$$(m+1)(m-4) = 0$$

$$m = -1, 4.$$

$$y = c_1 x^{-1} + c_2 x^4.$$

2. $4x^2 \cdot \frac{d^2y}{dx^2} + 8 \cdot x \cdot \frac{dy}{dx} + y = 0.$

3. $4x^2 \cdot y'' + 17y' = 0.$

2. $x^m\left(4 \cdot m^2 + (8-4)m + 1\right) = 0$

$$4m^2 + 4m + 1 = 0.$$

$$m^2 + m + \frac{1}{4} = 0$$

$$\left(m + \frac{1}{2}\right)^2 = 0 \Rightarrow m = -\tfrac{1}{2}, -\tfrac{1}{2}$$

$\therefore y = c_1 x^{-1/2} + c_2 x^{-1/2} \log x$

3. $4x^2 \cdot \frac{d^2y}{dx^2} + 17y = 0$

$4m^2 + (0-4)m.$

Q. $x^2 \cdot y'' + 9xy' - 20y = 0$. Put $x = e^t$ to transform the given cauchy-euler differential equation into linear differential equation with constant coefficients.

Q. $x^2y'' - 3xy' + 13y = 4 + 3x$

Legendre's linear differential equations

$$(ax+b)^2 \cdot \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = X$$

Place, $(ax+b) = e^t$

$\therefore t = \log(ax+b)$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt}\left(\frac{a}{ax+b}\right).$$

$$\Rightarrow (ax+b) \cdot \frac{dy}{dx} = a \cdot \frac{dy}{dt}$$

$$(ax+b)\frac{dy}{dx} = aDy$$

$D = \frac{d}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\cdot\frac{dt}{dx}\right) = \frac{d}{dx}\left(\underset{\substack{\downarrow \\ u}}{\frac{dy}{dt}}\cdot\underset{\substack{\downarrow \\ v}}{\frac{a}{ax+b}}\right)$$

$$\frac{d}{dx}(u\cdot v) = \cancel{u}\,\frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}$$

$$= \left(\frac{a}{ax+b}\right)\left(\frac{d}{dx}\cdot\frac{dy}{dt}\right) + \left(\frac{dy}{dt}\right)\cdot\frac{d}{dx}\left(\frac{a}{ax+b}\right)$$

$$\left(\frac{a}{ax+b}\right)\left(\frac{d}{dx}\cdot\frac{dy}{dt}\right) - \left(\frac{dy}{dt}\right)\cdot a\cdot\frac{a}{(ax+b)^2}$$

$$\left(\frac{a}{ax+b}\right)\left(\frac{d}{dx}\cdot\frac{dy}{dt}\right) - \left(\frac{dy}{dt}\right)\cdot\frac{a^2}{(ax+b)^2}$$

$$\cancel{\left(\frac{a}{ax+b}\right)\left(\frac{d}{dx}\left[\frac{dy}{dx}\cdot\frac{dx}{dt}\right]\right)} - \frac{dy}{dt}\cdot\frac{\cancel{a^2}}{(ax+\cancel{b})^2}$$

$$\left(\frac{a}{\cancel{ax+b}}\right)\left(\frac{dx}{dt}\,\cancel{\frac{d^2y}{dx^2}} + \cancel{\frac{dy}{dx}}\cdot\right)$$

$$\left(\frac{a}{ax+b}\right)\quad\frac{d}{dx}\left[\frac{dy}{dx}\cdot\frac{dt}{}\right.$$

Q. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$

$~~1+x = e^t~~$

Place $1+x = e^t$ $\quad\because a=1, b=1$

$(ax+b)\frac{dy}{dx} = aDy \Rightarrow (1+x)\frac{dy}{dx} = Dy$

$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$. & $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y + Dy + y = 2\sin t$

$D^2y - \cancel{Dy} + \cancel{Dy} + y = 2\sin t$

$$\boxed{(D^2+1)\,y = 2\sin t.}$$

Auxiliary equation: $m^2+1=0$

$\boxed{m = \pm i}$

C.F. $=$ ~~$c_1\cos^{\log(1+}$~~ $c_1\cos(\log(1+x)) + c_2\sin(\log(1+x))$.

P.I. $= \frac{2\sin t}{D^2+1}$

If $f(a)=0$,
P.I. $= t\cdot\frac{\sin t}{f'(D)}$

$D^2 = -1$

$\therefore$ P.I. $= \frac{2\sin t}{0}$

If $f'(a)=0$,
P.I. $= t^2\cdot\frac{\sin t}{f''(D)}$

$f'(D) = 2D$

$\therefore$ P.I. $= \frac{t\cancel{2}\sin t}{\cancel{2}D}$.

$P.I. = \frac{t}{} (-\cos t)$

$-\log(1+x) \cdot \cos t$

① $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Placing, $(2x+3) = e^t \Rightarrow x = \frac{e^t - 3}{2}$

$\Rightarrow t = \log(2x+3)$

$(2x+3)\frac{dy}{dx} = aDy = 2Dy$

$(2x+3)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y = 4D(D-1)y.$

$\therefore \quad 4D(D-1)y - 2Dy - 12y = 6x$

$y[4D^2 - 4D - 2D - 12] = 6x$

$y[4D^2 - 6D - 12] = 6x$

$y[2D^2 - 3D - 6] = 3x$

C.F. : $2m^2 - 3m - 6 = 0.$

$m = \frac{3 \pm \sqrt{9+48}}{4} = \frac{3 \pm \sqrt{57}}{4}$

$(c_1 t + c_2 t)e^t$

$c_1 e^{m_1 t} + c_2 e^{m_2 t}$

~~$y = c_1 \cos t + c_2 \sin t$~~ m

$c_1 \cancel{\cos}\left(\frac{3 + \sqrt{57}}{4}\right)t$

$c_1 e$

$$P.I. = \frac{3x}{2D^2-3D-6}.$$

$$= \frac{3.\left(\frac{e^t-3}{2}\right)}{2D^2-3D-6}.$$

$$= \frac{3e^t-9}{4D^2-6D-12}$$

$$\frac{3e^t}{4D^2-6D-12} \quad \frac{-9e^{0}}{4D^2-6D-12}$$

$$\frac{3e^t}{(4-6-12)} \quad \frac{-\cancel{9}3}{-\cancel{12}4}$$

$$\frac{3e^t}{-14} + \frac{3}{4}.$$

① $$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2.$$

$$m^3+3m^2+2m=0$$

$$m(m^2+3m+2)=0$$

$$m=0 \text{ \&}$$

$$m^2+2m+m+2=0$$

$$m(m+2)+1(m+2)=0$$

$$(m+1)(m+2)=0$$

$$m=-1,-2,0.$$

$$CF = c_1e^{0x}+c_2e^{-x}+c_3e^{-2x}.$$

$$P.I. = \frac{x^2}{D^3+3D^2+2D}$$

$$= \frac{x^2}{2D\left(1+\frac{D^3+3D^2}{2D}\right)}$$

$$= \frac{x^2}{2D}\left(1+\frac{D^2+3D}{2}\right)^{-1}$$

$$= \frac{x^2}{2D}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2 - \cdots\right]$$

$$= \frac{x^2}{2D}\left[1-\frac{D^2+3D}{2}+\frac{D^4+9D^2+6D^3}{4} - \cdots\right]$$

$$= \frac{1}{2D}\left[x^2-\frac{2}{2}-\frac{3\cdot 2x}{2}+\frac{0}{4}+\frac{9\cdot 2}{4 \cdot 2}+\frac{0}{4}\cdots\right]$$

$$= \frac{1}{2D}\left[x^2-1-3x+\frac{9}{2}\right]$$

$$= \frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]$$

$$P.I. = \frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7x}{2}\right]$$

$$\therefore C.S. = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7x}{2}\right]$$

Q $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 \cancel{+x+x}$

Auxiliary equation:

C.F.

$(m^2+2m+1) = 0$

$(m+1)^2 = 0$

$m = -1, -1.$

$y = c_1 e^{-x} + c_2 x e^{-x}$

$\frac{1}{1} e^{x}$

$\frac{1}{1} e^{-x}$

$P.I. = \frac{x^2}{D^2+2D+1}$

$= \frac{x^2}{1+(D^2+2D)}$

$= x^2 (1+(D^2+2D))^{-1}$

$= x^2 \left[1-(D^2+2D)+(D^2+2D)^2 - \cdots\right]$

$= x^2 - 2 \cancel{+} 4x + 4.2$

$P.I. = x^2 \cancel{+} 4x + 6.$

$\boxed{C.S. = c_1 e^{-x} + c_2 x e^{-x} + x^2 - 4x + 6.}$

Q $\frac{d^2y}{dx^2} + a^2 y = x \cos nx.$

C.F.

Auxiliary eq^n: $(m^2+a^2) = 0.$

$\therefore m^2 = -a^2$

$m = \pm\sqrt{-a^2}$ $\left\{\begin{matrix} a \pm ib \\ 0 \pm ai \end{matrix}\right\}$

$= \pm ai$

$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$= e^0 (c_1 \cos ax + c_2 \sin ax)$

$y = c_1 \cos ax + c_2 \sin ax$

$P.I. = \frac{x \cos nx}{D^2+a^2}$ $= R.P. \frac{e^{inx} \cdot x}{(D+in)^2+a^2}$

$$\frac{1}{D^2+2D}$$

$$2D\left[\frac{D}{2}+1\right]$$

R.P of $e^{inx} \cdot \frac{x}{D^2-n^2+2Din+a^2}$

$$(a^2-n^2)\left\{1+\frac{D^2+2Din}{a^2-n^2}\right\}$$

R.P. of $e^{inx} \cdot \frac{x}{D^2+a^2-n^2+2Din}$

R.P. of $e^{inx} \cdot \frac{x}{(a^2-n^2)\left[1+\frac{D^2+2Din}{a^2-n^2}\right]}$

$$\frac{x}{(a^2-n^2)}\left(1+\frac{D^2+2Din}{a^2-n^2}\right)^{-1}$$

$$\frac{x}{(a^2-n^2)}\left[1-\frac{D^2+2Din}{a^2-n^2}+\left(\frac{D^2+2Din}{a^2-n^2}\right)^2 \cdots\cdots\right]$$

$$\frac{1}{a^2-n^2}\left[x-\frac{0}{a^2-n^2}-\frac{2in}{a^2-n^2}\right]$$

R.P. of $e^{inx}\left(\frac{x}{a^2-n^2}-\frac{2in}{(a^2-n^2)^2}\right)$

R.P. of $(\cos nx+i\sin nx)\left(\frac{x}{(a^2-n^2)}-\frac{2in}{(a^2-n^2)^2}\right)$

R.P. of $\left\{\frac{x\cdot\cos nx}{a^2-n^2}-\frac{2n\cos nx\, i}{(a^2-n^2)^2}+\frac{x\sin nx\, i}{(a^2-n^2)}+\frac{2n\sin nx}{(a^2-n^2)^2}\right\}$

P.I. $=\frac{x\cdot\cos nx}{a^2-n^2}+\frac{2n\cdot\sin nx}{(a^2-n^2)^2}$

Method of variation of parameters:

$$a\frac{d^2y}{dx^2} + b\cdot\frac{dy}{dx} + cy = X$$

$C.F. = c_1y_1 + c_2y_2$

$P.I. = y = v_1y_1 + v_2y_2$; $v_1 = \int \frac{-y_2X}{W}$

$v_2 = \int \frac{y_1X}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Q $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

$(D^2+1)y = \operatorname{cosec} x$

$m^2+1=0 \Rightarrow m = \pm i$

$C.F. = \cancel{y^0} e^0 (c_1\cos x + c_2\sin x)$

$C.F. = c_1\underbrace{(\cos x)}_{y_1} + c_2\underbrace{(\sin x)}_{y_2}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$P.I. = v_1y_1 + v_2y_2$

$= v_1\cos x + v_2\sin x$

$$v_1 = \int -\frac{\cancel{\sin x}}{1}\cdot\frac{1}{\cancel{\sin x}}\,dx = -x$$

$$v_2 = \int \cancel{\sin x}\cdot \frac{\cos x}{1}\cdot\frac{1}{\sin x}\,dx$$

$$= \int \cot x\,dx = \ln|\sin x|$$

$P.I. = -x\cos x + \ln|\sin x|\sin x$

$C.S. = CF + P.I.$

$= c_1\cos x + c_2\sin x - x\cos x + \ln|\sin x|$

Q. $\frac{d^2y}{dx^2} + y = \sec x.$

$(m^2+1) = 0$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$m = \pm i$

$C.F. = C_1 \cos x + C_2 \sin x.$

$P.I. = V_1 \cos x + V_2 \sin x.$

V_1

Q $x^2 \cdot \frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} - y = x^2 e^x$

Dividing the equation by x^2,

$\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{x^2} = e^x$

$P = \frac{1}{x}$

$Q = \frac{-1}{x^2}$

check whether $P + Qx$ is 0 or not,

$\therefore \frac{1}{x} - \frac{1}{x^{\cancel{2}}} \cdot \cancel{x} = 0.$

If '0'. then,

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assume C.F = y = x.

Put $y = vx$.

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\cdot\frac{d^2v}{dx^2}.$$

Placing $\frac{d^2y}{dx^2}$

$$2\frac{dv}{dx} + x\cdot\frac{d^2v}{dx^2} + \frac{1}{x}\left(v + x\cdot\frac{dv}{dx}\right) - \frac{\cancel{x}\, v\cancel{x}}{x^2} = 0.$$

$$2\frac{dv}{dx} + x\cdot\frac{d^2v}{dx^2} + \cancel{\frac{v}{x}} + \frac{dv}{dx} - \cancel{\frac{v}{x}} = 0$$

$$x\cdot\frac{d^2v}{dx^2} + 3\frac{dv}{dx} = 0.$$

$$\frac{d^2v}{dx^2} = -\frac{3}{x}\cdot\frac{dv}{dx}$$

~~Integrating~~,

$$\int\frac{\frac{d^2v}{dx^2}}{\frac{dv}{dx}} = -\frac{3}{x} \qquad \int\frac{dt}{t} = \int -\frac{3}{x}$$

$$\ln t = -3\ln x$$

$$\ln\left|\frac{dv}{dx}\right| = \ln x^{-3}$$

$$\int\frac{dv}{dx} = \int -\frac{3}{x}$$

$$\ln\left|\frac{dv}{dx}\right| = \ln\frac{1}{x^3}$$

$$\int dv = \int -\frac{3}{x}\,dx \qquad \frac{dv}{dx} = \frac{1}{x^3}.$$

$$v = -3\int\frac{1}{x}\,dx \Rightarrow v = -3(\ln x) + c$$

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