Definition: (Point function) A point function $u = f(P)$ is a function that assigns some number or value u to each point P of some region R of space.

Examples: Scaler point function,
Vector point function

Scaler Point function: A scaler point function is a function that assigns a real number (i.e scaler) to each point of some region of space.

If to each point (x, y, z) of a region R in space there is assigned a real number $u = \phi(x, y, z)$, then ϕ is called a scaler point function.

Examples: 1. The temperature distribution within some body at a particular point in time.

Scaler field: A scaler point function defined over some region is called a scaler field.

Vector point function: A vector point function is a function that assigns a vector to each point of some region of space. If to each point (x, y, z) of a region R in space there is assigned a vector $F = F(x, y, z)$ then F is called a vector point function.

$$F = f_1(x, y, z)\,\hat{i} + f_2(x, y, z)\,\hat{j} + f_3(x, y, z)\,\hat{k}$$

Vector field: A vector point function defined over some region is called a vector field.

Level Surface: Let $f(x, y, z)$ be a single valued continuous scaler point function defined at every point $P \in D$.

Then
$$f(x, y, z) = c \quad (\text{Constant})$$
defines ~~a~~ the equation of a surface and is called a level surface of the function.

(*) For different values of c, we obtain different surfaces, no two of which intersect.

Exc: Find the Level surface of the scaler function.

$$f(x,y,z) = z - \sqrt{x^2+y^2}$$

Solution: Compute $f(x,y,z) = c$

$$\Rightarrow \quad z - \sqrt{x^2+y^2} = c$$

$$\Rightarrow \quad x^2 + y^2 = (z-c)^2$$

$\Rightarrow$ Level surfaces are Cones

Derivative of a vector function

Let $\vec{r} = \vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

be a vector function of a scaler variable t.

$$\frac{d\vec{r}}{dt} = \frac{d f_1(t)}{dt}\hat{i} + \frac{d f_2(t)}{dt}\hat{j} + \frac{d f_3(t)}{dt}\hat{k}$$

By definition

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}\hat{i} +$$

$$\lim_{\Delta t \to 0} \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}\hat{j} +$$

$$\lim_{\Delta t \to 0} \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t}\hat{k}$$

$$= \frac{d f_1(t)}{dt}\hat{i} + \frac{d f_2(t)}{dt}\hat{j} + \frac{d f_3(t)}{dt}\hat{k}$$

Properties:

$$\frac{d}{dt}\left(\vec{a} \pm \vec{b}\right) = \frac{d}{dt}(\vec{a}) \pm \frac{d}{dt}(\vec{b})$$

$$\frac{d}{dt}\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b}$$

$$\frac{d}{dt}\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \frac{d\vec{b}}{dt} + \vec{a} \times \frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}\left(\phi \vec{a}\right) = \phi \frac{d\vec{a}}{dt} + \frac{d\phi}{dt} \vec{a}$$

Vector Differential operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

- ∇ is known as Del operator
- ∇ operates on scaler field and produces a vector field.

Gradient of a scaler field function

If $\phi(x,y,z)$ is a scaler ~~field~~ point function, then

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\nabla\phi$ is called the gradient of scaler point function ϕ.

Problem: Find the gradient of scaler function

1. $f(x,y,z) = y^2 - 4xy$
2. $f(x,y,z) = x^2y^2 + xy^2 - z^2$ at $(3,1,1)$

Solution:

1. $f(x,y,z) = y^2 - 4xy$

$$\nabla f(x,y,z) = \frac{\partial}{\partial x}(y^2 - 4xy)\hat{i} + \frac{\partial}{\partial y}(y^2 - 4xy)\hat{j} + \frac{\partial}{\partial z}(y^2 - 4xy)\hat{k}$$

$$= -4y\hat{i} + (2y - 4x)\hat{j} + 0\hat{k}$$

2. $$\nabla f(x,y,z) = (2xy^2 + y^2)\hat{i} + (2x^2y + 2xy)\hat{j} - 2z\hat{k}$$

$$\nabla f(x,y,z)\Big|_{(3,1,1)} = 7\hat{i} + 24\hat{j} - 2\hat{k}$$

Example:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\vec{r}}{r}$$

Find $\nabla\left(\frac{1}{r}\right)$

Solution:

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right)$$

$$= \frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x)\,\hat{i}$$

$$-\frac{1}{2}\left(y^2 + x^2 + z^2\right)^{-3/2}(2y)\,\hat{j}$$

$$-\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2z)\,\hat{k}$$

$$= \frac{-\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{-\vec{r}}{r^3}$$

∇f is normal to the surface

- Let $f(x, y, z) = K$ be a level surface. consider a smooth curve c on the surface. Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be the parameteric representation of the curve c.
- The position vector of the point $P(x, y, z)$ on c is

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

- Since the curve c lies on the surface

$$\therefore\ f\big(x(t), y(t), z(t)\big) = K$$

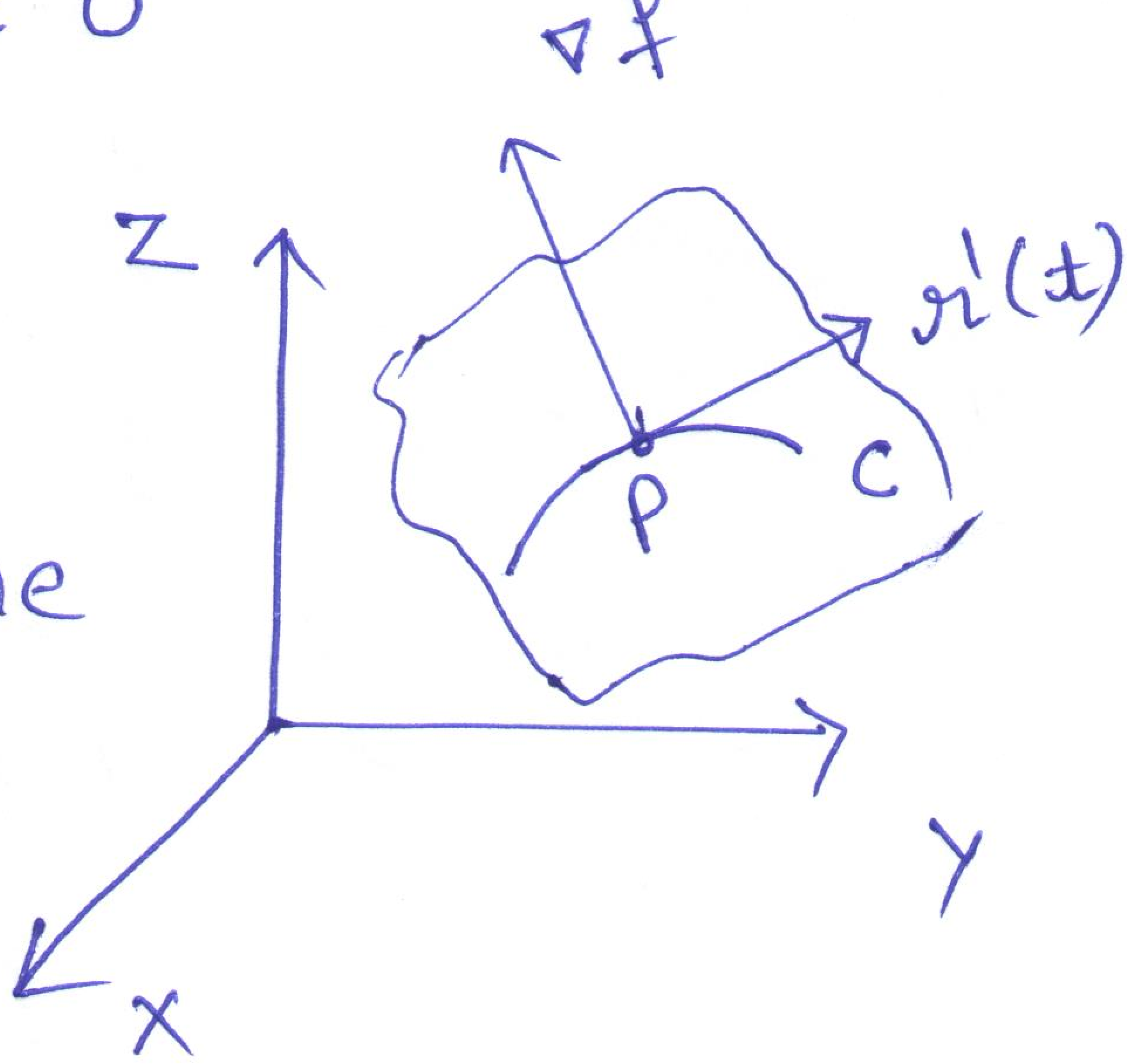
Then

$$\frac{d}{dt}\Big(f\big(x(t),\, y(t),\, z(t)\big)\Big) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dy}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$

$$\Rightarrow \quad \nabla f \cdot r'(t) = 0$$



$r'(t)$ is tangent vector to c at the point P and lies in the tangent plane to the surface at P.

Hence ∇f is orthogonal to every

tangent vector at P.

$\therefore$ ∇f is a vector normal to the surface $f(x, y, z) = K$ at the point P.

Example: Find the unit normal vector to the surface $xy^2 + 2yz = 8$ at $(3,-2,1)$

Solution:

$$f(x,y,z) = xy^2 + 2yz - 8$$

$$\nabla f = y^2\hat{i} + (2xy + z)\hat{j} + 2y\hat{k}$$

$$(\nabla f)_{(3,-2,1)} = 4\hat{i} - 10\hat{j} - 4\hat{k}$$

$$\hat{\nabla f} = \frac{4\hat{i} - 10\hat{j} - 4\hat{k}}{\sqrt{16 + 100 + 16}}$$

$$= \frac{2\hat{i} - 5\hat{j} - 2\hat{k}}{\sqrt{33}}$$

Exc: Find the unit normal vector to the surface $xy^2 - 2xyz = 3$ at $(1, 4, 3)$.

Solution: $\phi(x, y, z) = xy^2 - 2xyz = 3$

$$\nabla\phi = (y^2 - 2yz)\hat{i} + (2xy - 2xz)\hat{j} - 2xy\hat{k}$$

$$\nabla\phi\Big|_{(1,4,3)} = -8\hat{i} + 2\hat{j} - 8\hat{k}$$

Unit Normal vector

$$\hat{n} = \frac{(\nabla\phi)_{(1,4,3)}}{|\nabla\phi|}$$

$$= \frac{-8\hat{i} + 2\hat{j} - 8\hat{k}}{\sqrt{132}}$$

Directional Derivative

Definition: The Directional derivative of $f(x,y,z)$ in the direction of $\hat{b}$ is

$$\nabla f \cdot \hat{b}$$

where $f(x,y,z)$ is scaler field and $\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any unit vector.

Example: Find the Directional derivative of $f(x,y,z) = xy^2 + 4xyz + z^2$ at $(1,2,3)$ in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$.

Solution: $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$$\hat{b} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{5\sqrt{2}}$$

$$\nabla f = (y^2 + 4yz)\hat{i} + (2xy + 4xz)\hat{j} + (4xy + 2z)\hat{k}$$

$$(\nabla f)_{(1,2,3)} = 28\hat{i} + 16\hat{j} + 14\hat{k}$$

<u>Directional Derivative</u> $= \nabla f \cdot \hat{b}$

$$= \frac{(28\hat{i} + 16\hat{j} + 14\hat{k}) \cdot (3\hat{i} + 4\hat{j} - 5\hat{k})}{5\sqrt{2}}$$

$$= \frac{84 + 64 - 70}{5\sqrt{2}}$$

$$= \frac{78}{5\sqrt{2}}$$

<u>Exc:</u> Find the directional derivative of $f(x, y) = x^2y^3 + xy$ at $(2, 1)$ in the direction of a unit vector which makes an angle $\frac{\pi}{3}$ with x-axis.

<u>Solution:</u>

$$\nabla f = (2xy^3 + y)\,\hat{i} + (3x^2y^2 + x)\,\hat{j}$$

$$(\nabla f)_{(2,1)} = 5\hat{i} + 14\hat{j}$$

$$\begin{aligned}\hat{b} &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}\\ &= \cos\frac{\pi}{3}\,\hat{i} + \sin\frac{\pi}{3}\,\hat{j}\\ &= \frac{\hat{i}}{\sqrt{2}} + \frac{\sqrt{3}}{2}\,\hat{j}\end{aligned}$$

Directional Derivative $= \nabla f \cdot \hat{b}$

$$\begin{aligned}&= \left(5\hat{i} + 14\hat{j}\right)\cdot\left(\frac{\hat{i}}{\sqrt{2}} + \frac{\sqrt{3}}{2}\hat{j}\right)\\ &= \frac{5 + 14\sqrt{3}}{2}\end{aligned}$$

Note: Directional derivative of f in the direction of $\hat{b}$ is

$$\begin{aligned}D_b f = \nabla f \cdot \hat{b} &= |\nabla f|\,|\hat{b}|\cos\theta\\ &= |\nabla f|\cos\theta\end{aligned}$$

θ is the angle b/w the vectors ∇f and $\hat{b}$.

- $D_b f = |\nabla f|$ if $\theta = 0$

→ (Maximum value of $D_b f$)

→ $\hat{b}$ has the same direction of ∇f.

- $D_b f = -|\nabla f|$ if $\theta = \pi$

→ (Minimum value)

→ $\hat{b}$ and ∇f are in opposite direction.

<u>Conservative vector field:</u> A vector field $\vec{V}$ is said to be conservative if the vector function can be written as the gradient of a scaler function f, i.e,

$$\vec{V} = \nabla f$$

- Work done in conservative vector field ~~depends~~ is independent of path.

<u>Exc:</u> Show that the vector field defined by the vector function

$$\vec{V} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k}) \text{ is}$$

conservative.

$$\vec{V} = \nabla f$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = xy^2z^2 \quad \text{— (1)}$$

$$\frac{\partial f}{\partial y} = x^2yz^2 \quad \text{— (2)}$$

$$\frac{\partial f}{\partial z} = x^2y^2z \quad \text{— (3)}$$

Integrate (1) with respect to x

$$f(x, y, z) = \frac{1}{2}x^2y^2z^2 + g(y, z) \quad \text{— (4)}$$

from (2) & (4) $\quad x^2yz^2 + \frac{\partial g}{\partial y} = x^2yz^2$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 0 \Rightarrow g = g(z)$$

from (3) & (4)

$$x^2y^2z + \frac{\partial g}{\partial z} = x^2y^2z$$

$$\Rightarrow \frac{\partial g}{\partial z} = 0 \Rightarrow \quad g(z) = K = \text{Constant}$$

$\therefore \quad f(x, y, z) = \frac{1}{2} x^2 y^2 z^2 + K$

$\Rightarrow \quad \exists$ a scaler function $f(x, y, z)$

such that

$$\vec{V} = \nabla f$$

Hence $\vec{V}$ is a conservative field.

Divergence of a vector field $\vec{V}$

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

- If $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\text{div } \vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot \left(V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k} \right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

- $\nabla \cdot \vec{V}$ is just notation for div $\vec{V}$.
It is not a scaler product as

$$\nabla \cdot \vec{V} \neq \vec{V} \cdot \nabla$$

1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\operatorname{div} \vec{r} = 1 + 1 + 1 = 3$

2. If $\vec{V} = xyz\,\hat{i} + 3x^2 y\,\hat{j} + (xz^2 - y^2 z)\,\hat{k}$

Find $\nabla \cdot \vec{V}$ at $(2, -1, 1)$

$$\nabla \cdot \vec{V} = \left(yz + 3x^2 + 2xz - y^2\right)_{(2,-1,1)}$$

$$= -1 + 12 + 4 - 1$$

$$= 14$$

Curl of a vector point Function $\vec{V}$

$$\text{Curl } \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

Example 1. Prove that curl $\vec{r} = \vec{0}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{curl } \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

Example 2: $\vec{v} = xyz\,\hat{i} + 3x^2y\,\hat{j} + (xz^2 - y^2z)\,\hat{k}$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= -2yz\,\hat{i} + (xy - z^2)\,\hat{j} + (6xy - xz)\,\hat{k}$$

$$\left(\text{curl } \vec{v}\right)_{2,-1,1} = 2\hat{i} - 3\hat{j} - 14\hat{k}$$

Example: Prove that

$$\text{curl}(\text{grad } f) = 0$$

ie $\nabla \times (\nabla f) = 0$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{i} + \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

Prove that $\nabla \cdot (\nabla \times \vec{V}) = 0$

Solution: If $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \hat{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_2}{\partial x} \right) \hat{j}$$

$$+ \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \hat{k}$$

$$\nabla \cdot (\nabla \times \vec{V}) = \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$= 0$$

Example: Prove that

$$\operatorname{div}(f\vec{V}) = f(\operatorname{div}\vec{V}) + \nabla f\cdot\vec{V}$$

$f(x, y, z)$ is a scaler point function.

$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is a vector function.

Solution:

$$\nabla\cdot(f\vec{V}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot$$

$$\left(f V_1\hat{i} + f V_2\hat{j} + f_3 V_3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(fV_1) + \frac{\partial}{\partial y}(fV_2) + \frac{\partial}{\partial z}(fV_3)$$

$$= f\frac{\partial V_1}{\partial x} + \frac{\partial f}{\partial x}V_1 + \frac{\partial f}{\partial y}V_2 + f\frac{\partial V_2}{\partial y}$$

$$+ \frac{\partial f}{\partial z}V_3 + f\frac{\partial V_3}{\partial z}$$

$$= f\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + \left(\frac{\partial f}{\partial x}V_1 + \frac{\partial f}{\partial y}V_2 + \frac{\partial f}{\partial z}V_3\right)$$

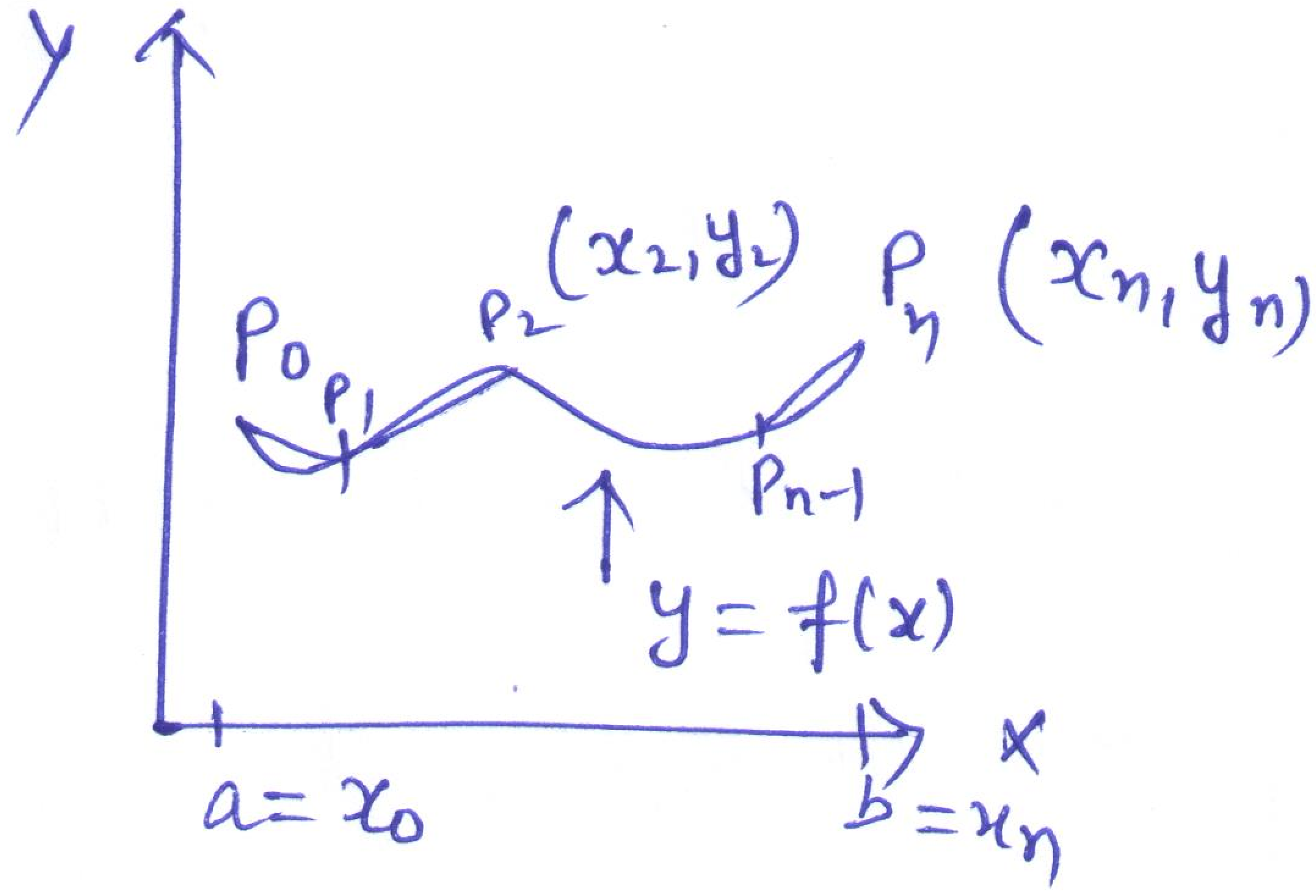
$$= f\,\nabla\cdot\vec{V} + \nabla f\cdot\vec{V}$$

Remark:

1. If $\operatorname{div}\vec{V} = 0$, then the vector field $\vec{V}$ is said to be Solenoidal.

2. If $\operatorname{curl}\vec{V} = 0$, then the vector field $\vec{V}$ is said to be irrotational.

Recall: (Arc Length)



Length of the curve

$$L \simeq \sum_{i=1}^{n} |P_{i-1} P_i|$$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i| \quad -(1)$$

: $P_{i-1}(x_{i-1}, y_{i-1})$

: $P_i(x_i, y_i)$

$$|P_{i-1} P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \sqrt{\Delta x^2 + [f'(x_i^*)]^2 \Delta x^2} \quad - (*)$$

$$= \sqrt{1 + f'(x_i^*)^2} \quad \Delta x$$

{ (*) By Mean value on $[x_{i-1}, x_i]$

$\exists \; x_i^* \in (x_{i-1}, x_i)$ such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

$$\Delta y_i = f'(x_i^*) \Delta x \quad \}$$

$\therefore$ (1) becomes

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2} \, \Delta x$$

$$L = \int_a^b \sqrt{1 + f'(x)^2} \, dx$$

- $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

- $$L = \int_a^b ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{if } y = f(x)$$

$$a \leq x \leq b$$

or

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{if } x = h(y)$$

$$c \leq y \leq d$$

Exc: Determine the length of the curve $y = \ln(\sec x)$, $0 \le x \le \frac{\pi}{4}$

Solution:

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln(\sec x + \tan x)$$

$$= \ln(\sqrt{2} + 1)$$

Exc: Determine the arc length of

$$x = \frac{2}{3}(y-1)^{3/2} \; ; \; 1 \le y \le 4$$

$$L = \int_1^4 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + y - 1}\, dx = \left. \frac{y^{3/2}}{3/2} \right|_1^4 = \frac{14}{3}$$

Line Integral: Integral which is evaluated along a curve c.

1. Line integral of $f(x,y)$ along c is

$$\int_C f(x,y)\, ds \quad \text{(scaler field)}$$

$$= \int_C f(x,y) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

if $y = g(x)$, $a \le x \le b$

$$= \int_C f(x,y) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

if $x = g(y)$

$c \le y \le d$

2. $$\int_C f(x,y)\, ds = \int_a^b f(h(t), g(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

$$\left.\begin{array}{l} x = h(t) \\ y = g(t) \end{array}\right\} \text{parametric form}$$

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j}, \quad a \le t \le b$$

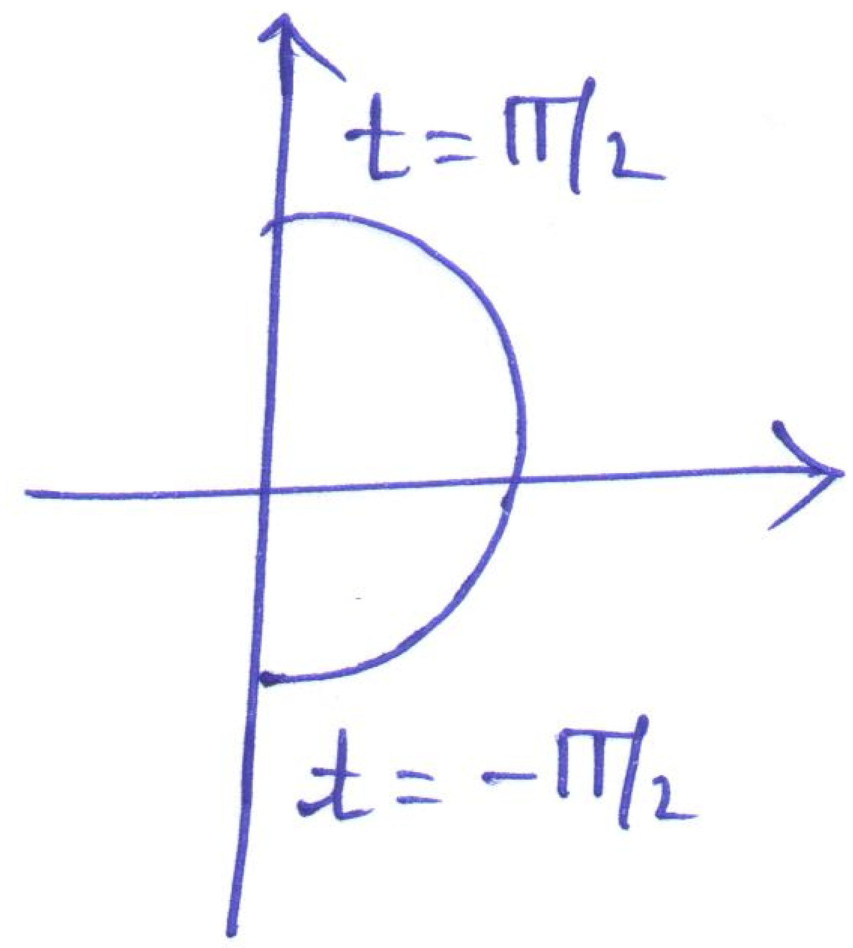
$$= \int_a^b f(h(t), g(t)) \| r'(t) \| \, dt$$

$$\| r'(t) \| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Note: It is computationally easy to evaluate integral in parametric form.

Evaluate: $\int_C x y^4 \, ds$,

Where C is the right half of the circle $x^2 + y^2 = 16$ traced out in a Counter clockwise direction.



Solution:

$$x = 4 \cos t$$
$$y = 4 \sin t$$

$$\frac{dx}{dt} = -4 \sin t$$

$$\frac{dy}{dt} = 4 \cos t$$

$$\| r'(t) \| = 4$$

$$\int_C x y^4 \, ds = \int_{-\pi/2}^{\pi/2} (4 \cos t)(4 \sin t)^4 \cdot 4 \, dt$$

$$= 4096 \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t \, dt$$

$$= \frac{8192}{5}$$

Remark: In three variables x, y, z

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \frac{ds}{dt}\, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \|\vec{r}'(t)\|\, dt$$

Evaluate $\int_C xyz\, ds$, $\vec{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j} + 3t\, \hat{k}$
$0 \leq t \leq 4\pi$

$$\int_C xyz\, ds = \int_0^{4\pi} 3t \cos t \sin t \sqrt{\sin^2 t + \cos^2 t + 9}\, dt$$

$$= 3 \int_0^{4\pi} t\left(\frac{1}{2} \sin 2t\right) \sqrt{10}\, dt$$

$$= -3\sqrt{10}\, \pi$$

Evaluate: $\int_C x^2 y \, ds$, where C is the curve defined by $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \frac{\pi}{2}$

Solution: $ds = \frac{ds}{dt} dt$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \sqrt{9} \, dt$$

$$= 3 \, dt$$

$$\int_C x^2 y \, ds = \int_0^{\frac{\pi}{2}} 27 \cos^2 t \sin t \, dt$$

$$= 27$$

Line Integral of Vector field $\vec{F}$ along C

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt$$

Evaluate: $\int_C \vec{F}\cdot d\vec{r}$

$$\vec{F} = 8x^2yz\,\hat{i} + 5z\,\hat{j} - 4xy\,\hat{k}$$

$$C: \vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}, \quad 0 \le t \le 1$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(8x^2yz\,t + 5zt^2 - 4xy\,t^3\right) dt$$

$$= \int_0^1 8t^7 + 10t^4 - 12t^5 \, dt$$

$$= \left. t^8 + 2t^5 - 2t^6 \right]_0^1$$

$$= 1$$

Exc: If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution: $y = 2x^2$

Parametric equations are

$$x = t \qquad 0 \le t \le 1$$
$$y = 2t^2$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 3xy\,\frac{dx}{dt} - y^2\,\frac{dy}{dt}$$

$$= \int_0^1 \left(3t\cdot 2t^2\cdot 1 - 4t^4\cdot 4t\right) dt$$

$$= \int_0^1 \left(6t^3 - 16t^5\right) dt = \frac{-7}{6}$$

Work done: The work done by force $\vec{F}$ during displacement from A to B is given by

$$\int_A^B \vec{F}\cdot d\vec{r}$$

Note: If $\vec{F}$ is a conservative vector field, then

$\vec{F} = \nabla f$, $f(x,y,z)$ is a scaler field.

$$\text{Work done} = \int_P^Q \vec{F}\cdot d\vec{r}$$

$$= \int_P^Q \nabla f \cdot d\vec{r}$$

$$= \int_P^Q \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

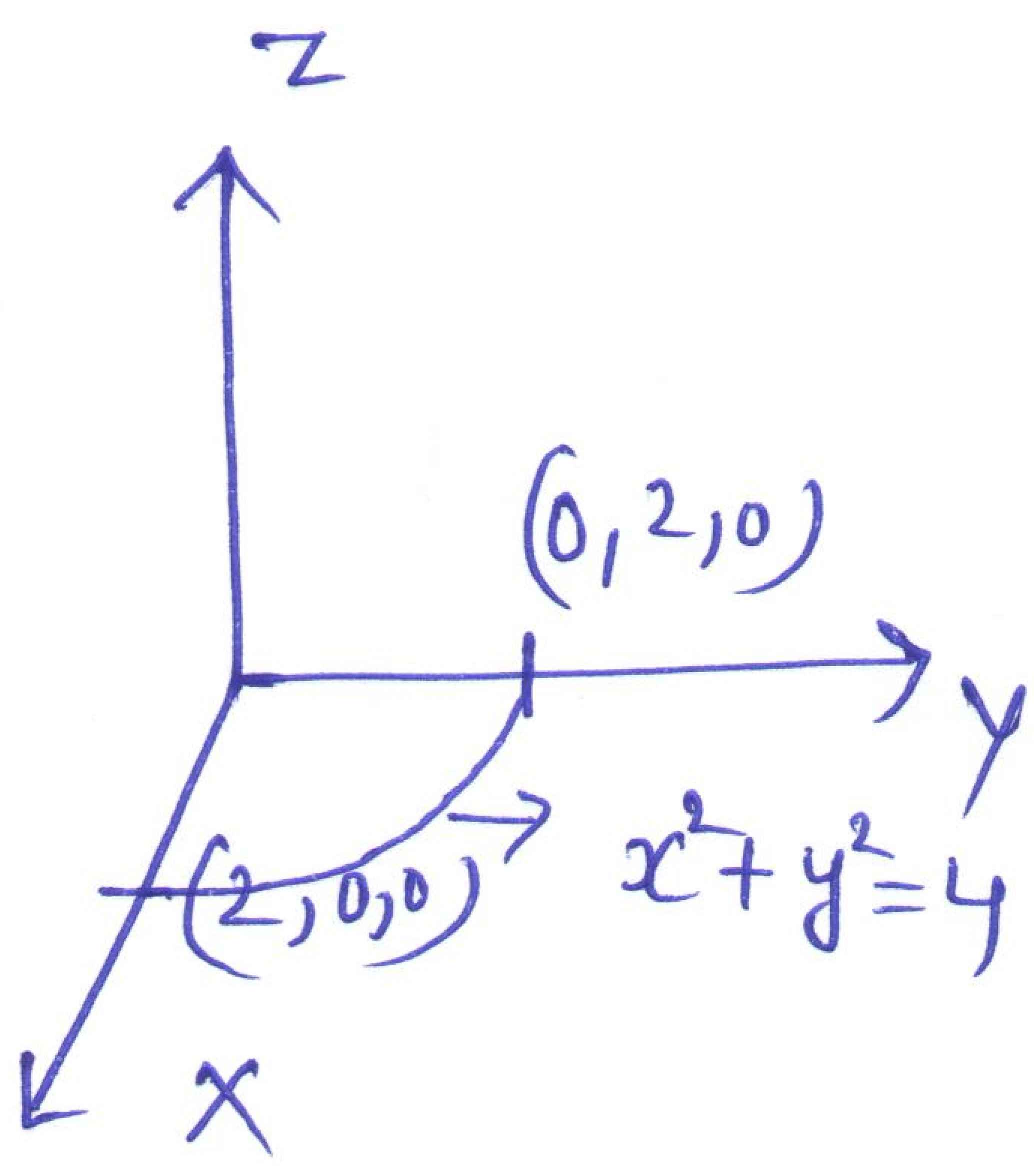
$$= \left[f(x,y,z)\right]_P^Q$$

Example: Find the work done by the force

$$\vec{F} = -xy\,\hat{i} + y^2\,\hat{j} + z\,\hat{k}$$

in moving a particle over the circular path $x^2+y^2=4$, $z=0$ from $(2,0,0)$ to $(0,2,0)$

Solution: The parametric representation of the given curve is



$x = 2\cos t$

$y = 2\sin t$

$z = 0$, $0 \le t \le \frac{\pi}{2}$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C -xy\,dx + y^2\,dy + z\,dz$$

$$= \int_0^{\frac{\pi}{2}} -4\sin t\cos t\,(-2\sin t) + 4\sin^2 t\,(2\cos t)\cdot dt$$

$$= 16\int_0^{\pi/2} \sin^2 t\cos t\,dt = \frac{16}{3}$$

Example: Show that the vector field $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ is Conservative. Find its scaler potential and the work done in moving a particle from $(-1,2,1)$ to $(2,3,4)$.

Solution: $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix}$$

$$= 0$$

$\Rightarrow \vec{F}$ is conservative

$\Rightarrow \vec{F} = \nabla f,$

Hence, $\frac{\partial f}{\partial x} = 2x(y^2+z^3) \quad -(1)$

$$\frac{\partial f}{\partial y} = 2x^2 y \quad -(2)$$

$$\frac{\partial f}{\partial z} = 3x^2 z^2 \quad -(3)$$

Integrate (1) w.r.t x

$$f(x,y,z) = x^2(y^2+z^3) + g(y,z) \quad -(4)$$

Differentiate w.r.t y

$$\frac{\partial f}{\partial y} = 2x^2 y + \frac{\partial g}{\partial y} \quad -(5)$$

from (2) & (5), we have

$$\frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$$

$$\therefore \quad f(x,y,z) = x^2(y^2+z^3) + h(z) \quad -(6)$$

Differentiate (6) w.r.t z

$$\frac{\partial f}{\partial z} = 3x^2 z^2 + \frac{dh}{dz} \quad -(7)$$

from (3) & (7) $\Rightarrow \frac{dh}{dz} = 0 \Rightarrow h(z) = c$

$\therefore$ Equation (4) is $f(x,y,z) = x^2(y^2+z^3) + c$

Work done by $\vec{F}$ in moving a particle $P(-1,2,1)$ to $Q(2,3,4)$ is

$$W = \int_P^Q \vec{F} \cdot d\vec{r}$$

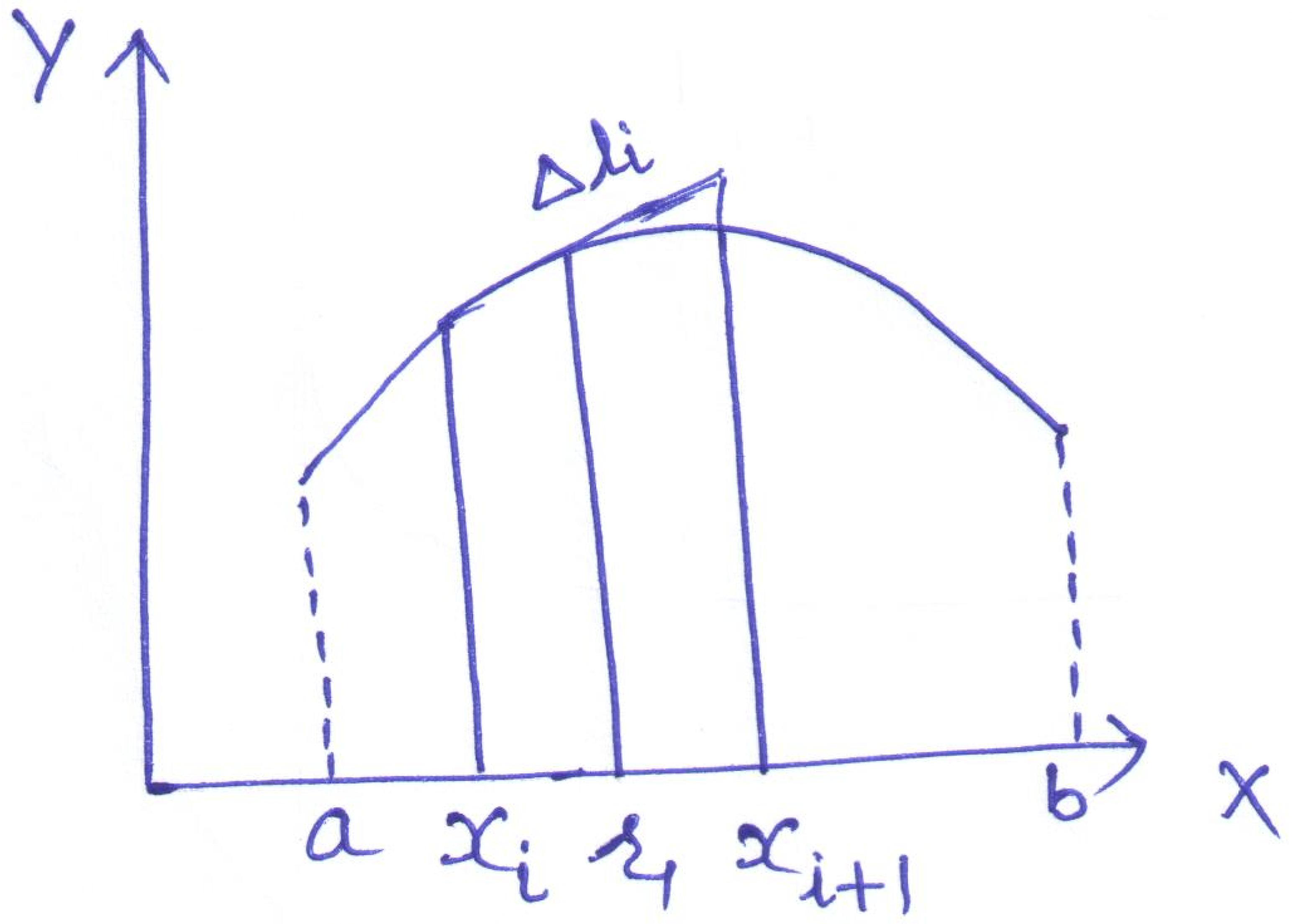
$$= \left[f(x,y,z) \right]_P^Q$$

$$= \left[x^2 (y^2 + z^3) \right]_{(-1,2,1)}^{(2,3,4)}$$

$$= 287$$

Computation of Surface Area

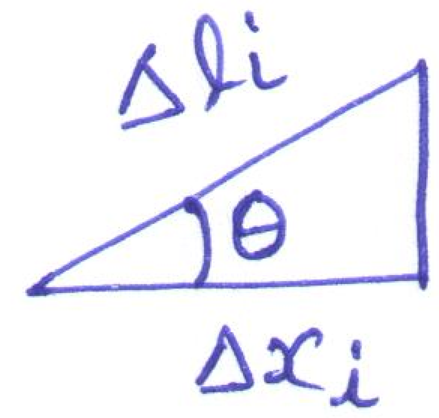
Recall: Computation of curve length



Length of the curve

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta l_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1+(f'(\xi_i))^2}\,\Delta x_i$$



$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta$$

$$\Rightarrow \Delta l_i = \frac{1}{\cos\theta}\,\Delta x_i$$

$$\text{Hy } \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\Rightarrow \frac{1}{\cos\theta} = \sqrt{1+(f'(\xi_i))^2}$$

$$\Rightarrow \Delta l_i = \sqrt{1+(f'(\xi_i))^2}\,\Delta x_i$$

$$\Rightarrow L = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Computation of Surface Area

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Where D is the projection of the surface in the xy-plane.

Similarly, if the equation is given in the form $x = u(y, z)$ or in the form $y = \phi(x, z)$, then

$$S = \iint_{\hat{D}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

or

$$S = \iint_{\hat{\hat{D}}} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz$$

Where $\hat{D}$ and $\hat{\hat{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

Problem: Compute the Surface area of the sphere $x^2+y^2+z^2=a^2$.

Solution: Equation of the Surface

$$z=\sqrt{a^2-x^2-y^2} \quad \text{(upper Half)}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

Domain of integration $x^2+y^2 \le a^2$

$$S = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dy\, dx$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2-y^2}}\, dy\, dx$$

$$= 2\int_{0}^{2\pi}\int_{0}^{a} \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta = 4\pi a^2$$

Exc: Determine the surface area of the part $z = xy$ that lies in the cylinder $x^2 + y^2 = 1$

Solution: $z = f(x,y) = xy$

$$z_x = y$$

$$z_y = x$$

$$S = \iint_D \sqrt{1 + x^2 + y^2}\, dx\, dy$$

In polar coordinates

$$S = \int_0^{2\pi} \int_{r=0}^{1} \sqrt{1+r^2}\; r\, dr\, d\theta$$

$$= 2\pi \cdot \left[\frac{(1+r^2)^{3/2}}{2 \cdot 3/2} \right]_0^1$$

$$= \frac{2\pi}{3} \left(2^{3/2} - 1 \right)$$

Exc: Find the area of that part of the sphere $x^2+y^2+z^2=a^2$ that is cut off by the cylinder

$x^2+y^2=ax$

Solution: $z=\sqrt{a^2-x^2-y^2}$

$$z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}$$

$$z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$S=2\cdot 2\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$=4\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$\Rightarrow 4 \iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$$

$$= 4 \int_0^{\pi/2} \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= 4a \int_0^{\pi/2} \left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a \int_0^{\pi/2} [-a\sin\theta + a]\, d\theta$$

$$= 4a\left[a\cos\theta + a\theta\right]_0^{\pi/2}$$

$$= 2a^2(\pi - 2)$$

Green's Theorem: Let C be a smooth simple closed curve bounding a region R. If f, g, $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial x}$ are continuous on R, then

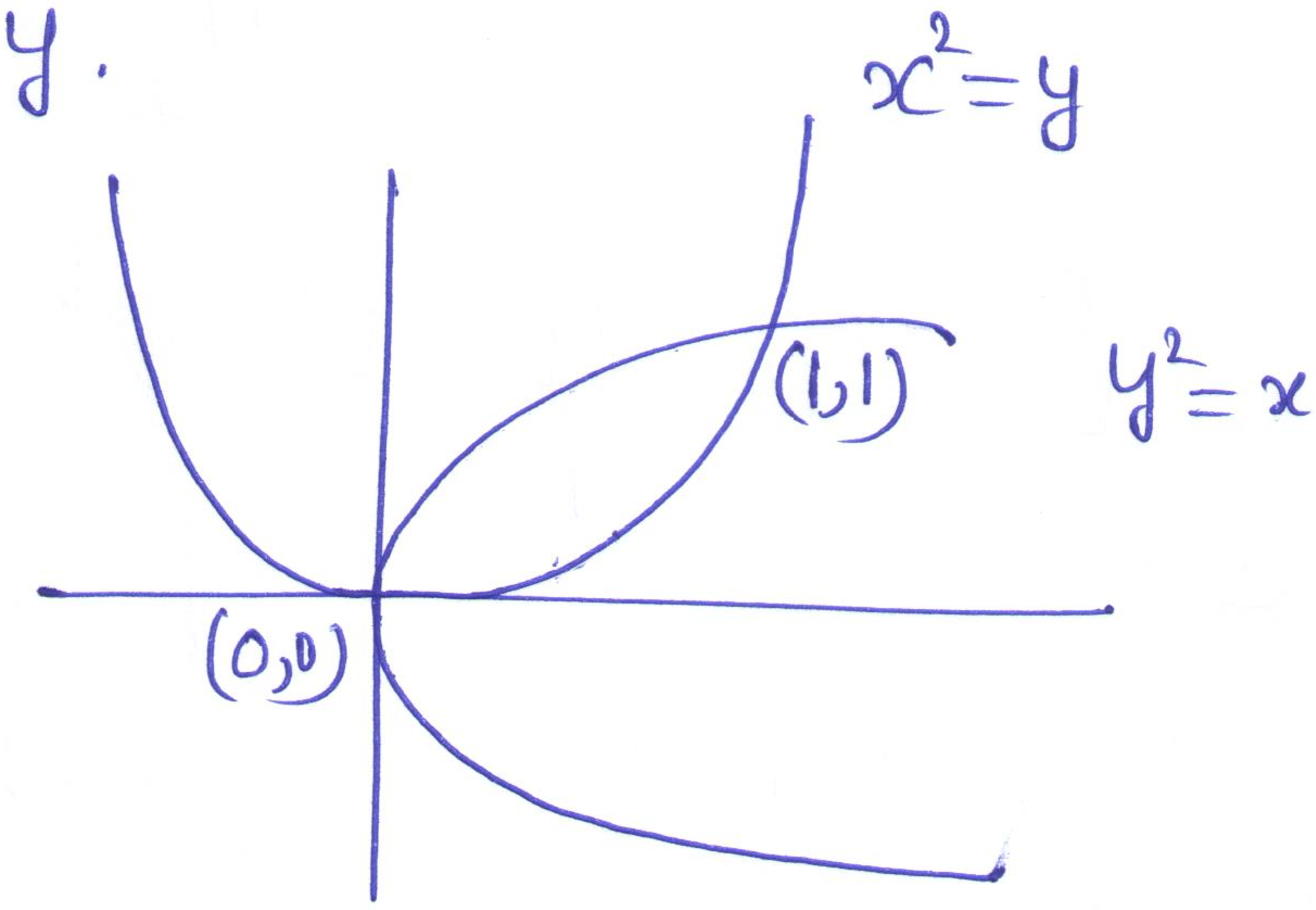
$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy \quad \text{— (1)}$$

The integration being carried in the positive direction (anticlockwise direction) of C.

Note: Green's theorem provides a relationship between a double integral over a region R and the line integral over the closed curve C bounding R.

(285)

Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$, where C is the boundary of the region in the first quadrant that is bounded by the curves $y^2 = x$ and $x^2 = y$.



Solution:

$f(x,y) = x^2 + y^2$

$g(x,y) = y + 2x$

$\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = 2$

Using Green's theorem

$$\oint_C (x^2+y^2)\,dx + (y+2x)\,dy = \iint_R (2-2y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-2y)\,dy\,dx = \frac{11}{30}$$

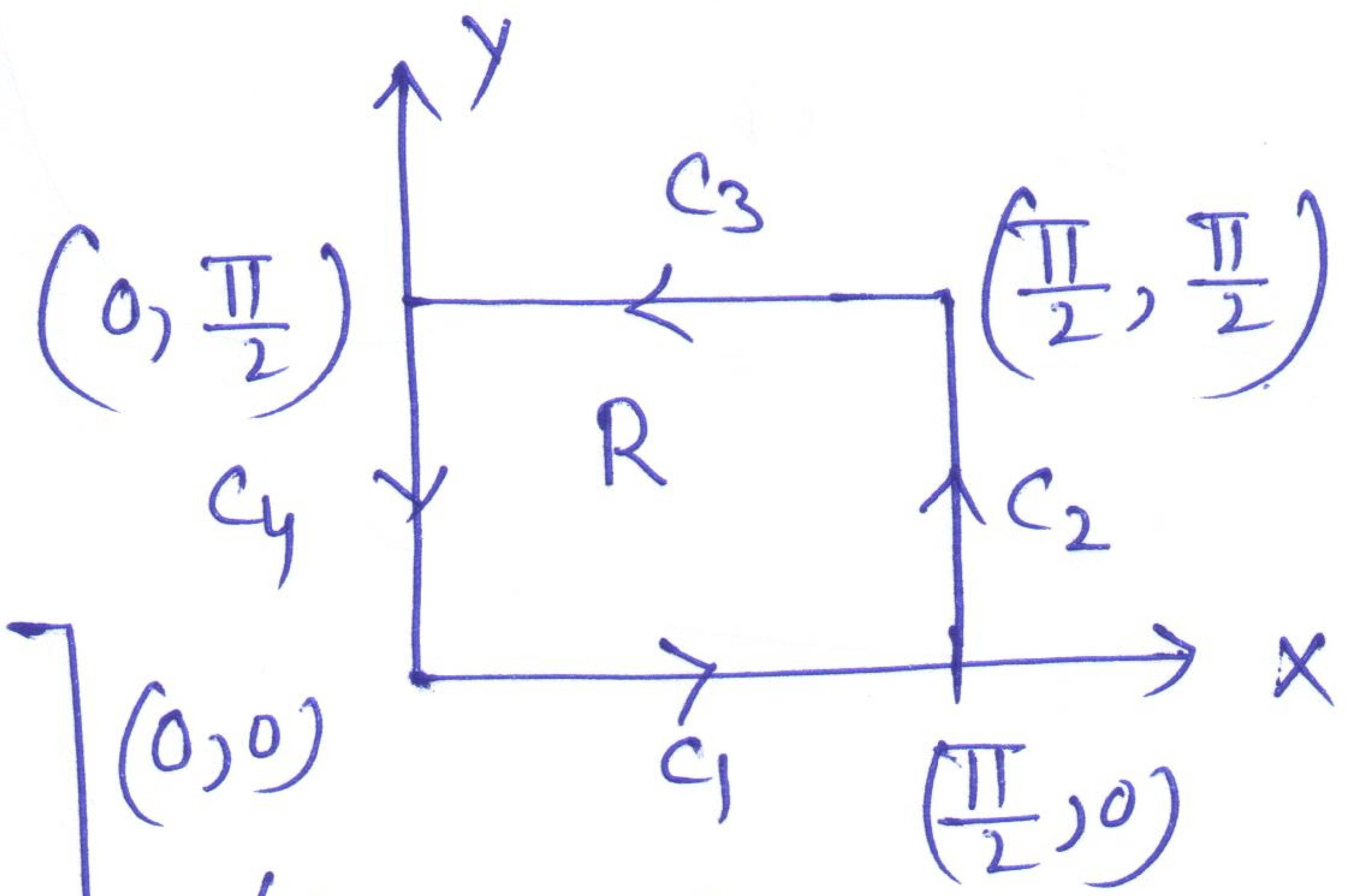
Exc: Verify the Green's theorem

$$f(x, y) = e^{-x} \sin y$$

$$g(x, y) = e^{-x} \cos y \quad \text{and}$$

C is square with vertices at $(0,0)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$

$$\oint_C f(x,y)\,dx + g(x,y)\,dy$$



$$= \left[\int_{C_1} + \int_{C_1} + \int_{C_3} + \int_{C_4}\right] \cdot (f\,dx + g\,dy)$$

Where C_1, C_2, C_3, C_4 are boundary lines.

Along C_1: $\quad y = 0, \quad 0 \leq x \leq \frac{\pi}{2}$

$$\int_{C_1} e^{-x} \sin y\,dx + e^{-x} \cos y\,dy = 0$$

Along C_2: $x = \frac{\pi}{2}$, $0 \le y \le \frac{\pi}{2}$

$$\int_{C_2} e^{-x}(\sin y\, dx + \cos y\, dy)$$

$$= \int_0^{\pi/2} e^{-\pi/2} \cos y\, dy = e^{-\pi/2}$$

Along C_3: $y = \frac{\pi}{2}$, $\frac{\pi}{2} \le x \le 0$

$$\int_{C_3} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_{\frac{\pi}{2}}^{0} e^{-x}\, dx$$

$$= e^{-\pi/2} - 1$$

Along C_4 $x = 0$, $\frac{\pi}{2} \le y \le 0$

$$\int_{C_4} e^{-x}(\sin y\, dx + \cos y\, dy) = -1$$

$$\oint_C f\,dx + g\,dy = e^{-\pi/2} + e^{-\pi/2} - 1 - 1 = 2(e^{-\pi/2} - 1)$$

Using Green's theorem

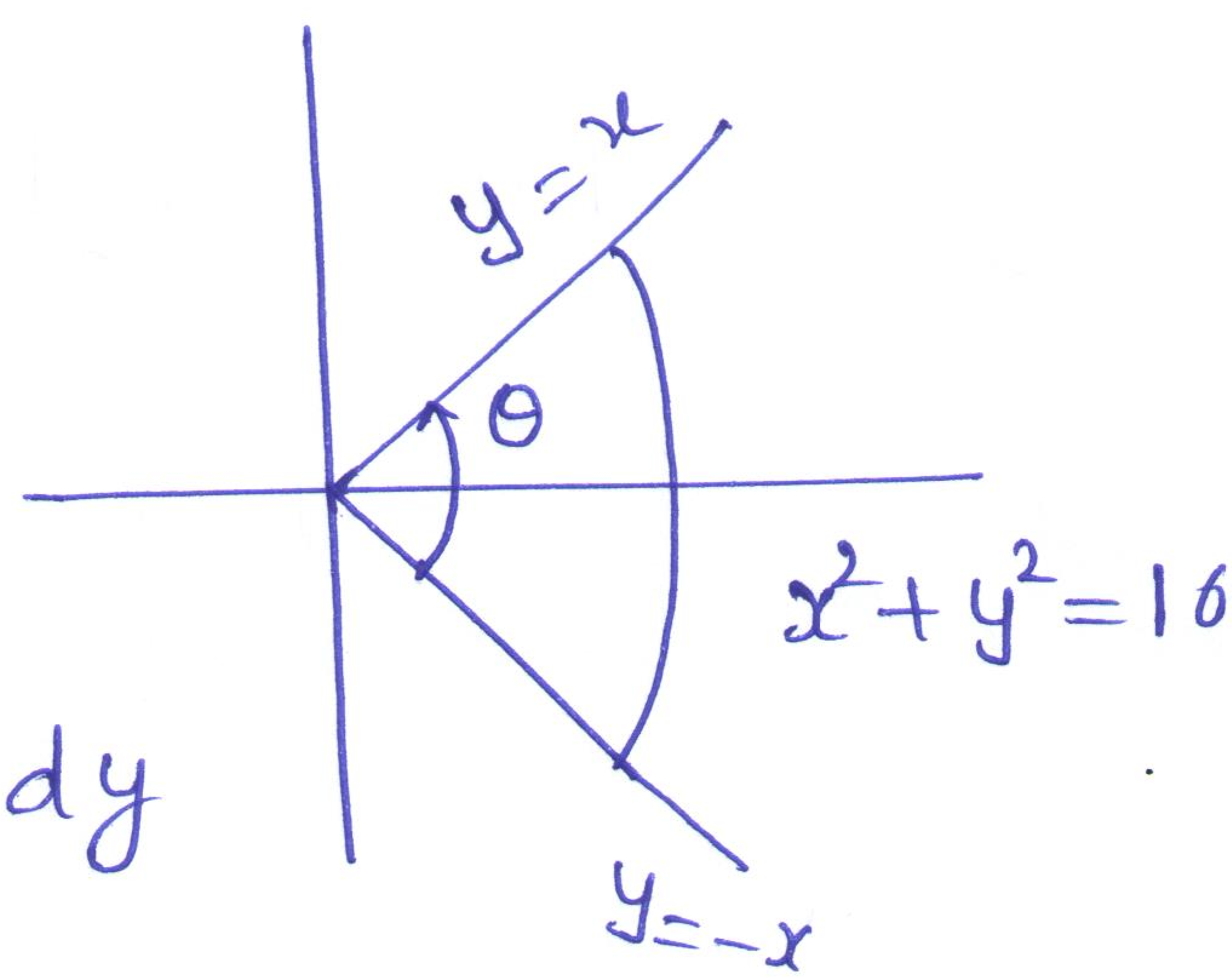
$$\oint_C f\,dx + g\,dy = \iint_R -2e^{-x} \cos y\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} -2e^{-x}\, dx\, dy$$

$$= 2\left(e^{-\pi/2} - 1\right)$$

Exc: Find the work done by the force $\vec{F} = (x^2 - y^3)\,\hat{i} + (x+y)\,\hat{j}$ in moving a particle along the closed curve C containing curves $x+y=0$, $x^2+y^2=16$, $y=x$. in the first and fourth quadrant.

Solution:



$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C (x^2 - y^3)\, dx + (x+y)\, dy$$

$$= \iint_R (1 + 3y^2)\, dx\, dy \quad \text{(Green's Theorem)}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\iint_R (1+3y^2)\,dx\,dy = \int_{-\pi/4}^{\pi/4}\int_0^4 (1+3r^2\sin^2\theta)\,r\,dr\,d\theta$$

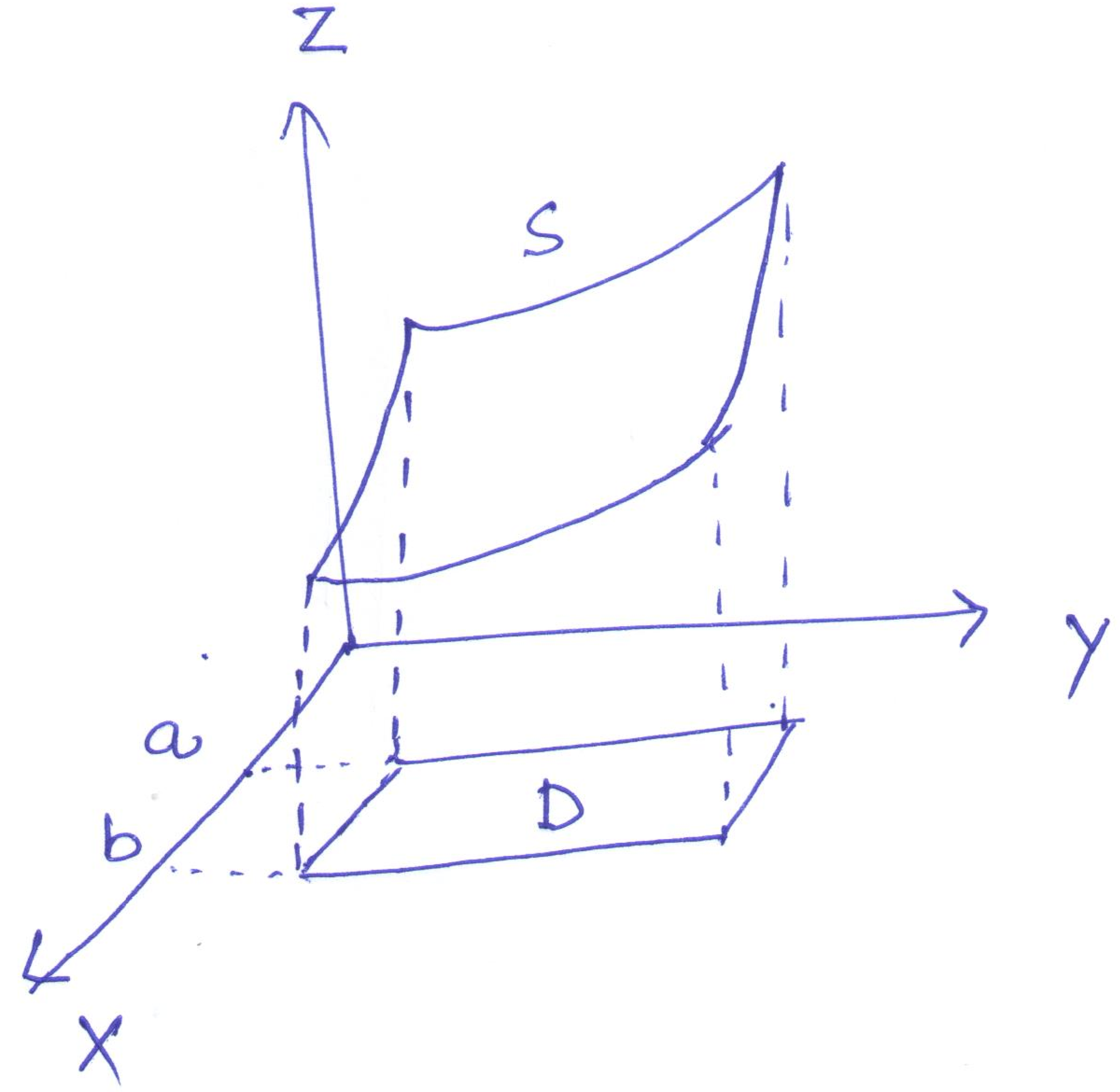
$$= \int_{-\pi/4}^{\pi/4} \left(\frac{r^2}{2} + \frac{3r^4}{4}\sin^2\theta\right)_0^4 d\theta$$

$$= \int_{-\pi/4}^{\pi/4} (8 + 192\sin^2\theta)\,d\theta$$

$$= 2\int_0^{\pi/4} 8 + 96(1-\cos 2\theta)\,d\theta$$

$$= 52\pi - 96$$

Surface Integral of Scaler field



1. $$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y)) \cdot \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

$S: \; Z = g(x,y)$ $\quad (dA = dx\,dy)$

2. $S: \; x = h(y,z)$

$$\iint_S f(x,y,z)\,ds = \iint_D f(h(y,z), y, z) \cdot \sqrt{1 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}\; dy\,dz$$

3. $S: y = \phi(x,z)$

$$\iint_S f(x,y,z)\, dz = \iint_D f(x, \phi(x,z), z) \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + 1 + \left(\frac{\partial \phi}{\partial z}\right)^2} \cdot dx\, dz$$

4. Parametric form

$$\iint_S f(x,y,z)\, ds = \iint_D f(x(u,v), y(u,v), z(u,v)) \cdot \|\vec{r}_u \times \vec{r}_v\|\, dA$$

Note:

1 $\Leftrightarrow$ 4. $z = g(x,y)$

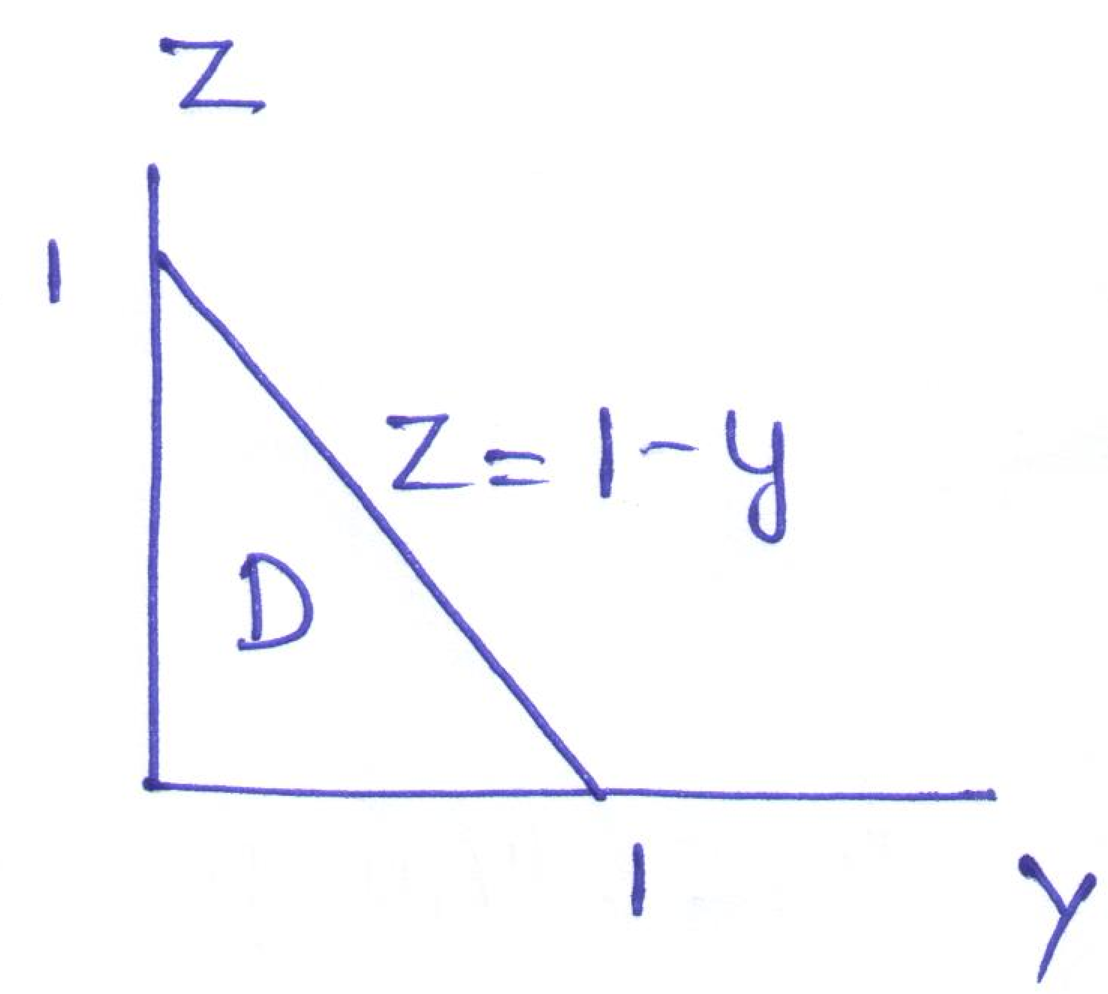
$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + g(x,y)\hat{k}$$

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

Example: Evaluate

$\iint_S 6xy\,ds$: Where S is the portion of the plane $x+y+z=1$ that lies in the first octant and is in front of yz-plane.

Solution: $x = 1-y-z$
$= g(y,z)$



$$\iint_S f(x,y,z)\,ds$$

$$= \iint_D f(g(y,z),y,z)\sqrt{1+\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2}\cdot dy\,dz$$

$$= \iint_D 6(1-y-z)y\sqrt{1+(-1)^2+(-1)^2}\,dy\,dz$$

$$= \sqrt{3}\iint_D 6y(1-y-z)\,dA$$

$$= \sqrt{3} \int_0^1 \int_0^{1-y} 6y(1-y-z)\, dz\, dy$$

$$= \frac{\sqrt{3}}{4}$$

Example: $\cancel{\iiint_E z\, dE}$ $\iint_S z\, ds$

S: upper half of the sphere of radius 2.

$$\vec{r}(\theta, \phi) = 2\sin\phi\cos\theta\,\hat{i} + 2\sin\phi\sin\theta\,\hat{j} + 2\cos\phi\,\hat{k}$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}$$

$$\vec{r}_\theta(\theta, \phi) = -2\sin\phi\sin\theta\,\hat{i} + 2\sin\phi\cos\theta\,\hat{j} + 0\hat{k}$$

$$\vec{r}_\phi(\theta, \phi) = 2\cos\phi\cos\theta\,\hat{i} + 2\cos\phi\sin\theta\,\hat{j} - 2\sin\phi\,\hat{k}$$

$$\vec{r}_\theta \times \vec{r}_\phi = -4\sin^2\phi\cos\theta\,\hat{i} - 4\sin^2\phi\sin\theta\,\hat{j} - 4\sin\phi\cos\phi\,\hat{k}$$

$$\| \vec{r}_\theta \times \vec{r}_\phi \| = 4 \sin\phi$$

$$\iint_S z\, ds = \iint_D 2\cos\phi \, (4\sin\phi) \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 8 \cos\phi \sin\phi \, d\theta \, d\phi$$

$$= 8\pi$$

<u>Example:</u> Evaluate $\iint_S y\, ds$

S: Portion of the cylinder that lies between $z = 0$ and $z = 6$.

$$\vec{r}(z, \theta) = \sqrt{3}\cos\theta \, \hat{i} + \sqrt{3}\sin\theta \, \hat{j} + z\, \hat{k}$$

$$0 \le z \le 6, \quad 0 \le \theta \le 2\pi$$

$$\vec{r}_z(z, \theta) = \hat{k}$$

$$\vec{r}_\theta(z, \theta) = -\sqrt{3}\sin\theta \, \hat{i} + \sqrt{3}\cos\theta \, \hat{j}$$

$$\vec{r}_z \times \vec{r}_\theta = -\sqrt{3}\cos\theta\,\hat{i} - \sqrt{3}\sin\theta\,\hat{j}$$

$$\| \vec{r}_z \times \vec{r}_\theta \| = \sqrt{3}$$

$$\iint_S y\,dS = \iint_D \sqrt{3}\sin\theta \cdot \sqrt{3}\,dA$$

$$= 3\int_0^{2\pi}\int_0^{6} \sin\theta\,dz\,d\theta$$

$$= 0$$

Surface Integral of vector field

Surface Integral of $\vec{F}$ over surface S is

$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\,ds$$

$\hat{n}$ is unit normal vector to the surface S.

- $S: \; Z = g(x, y)$

$$f(x, y, z) = z - g(x, y)$$

$$\hat{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{-g_x\hat{i} - g_y\hat{j} + \hat{k}}{\left[(g_x)^2 + (g_y)^2 + 1\right]^{\frac{1}{2}}}$$

$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\,ds =$$

$$ds = \sqrt{(g_x)^2 + (g_y)^2 + 1}\; dA = \|\nabla f\|\, dA$$

$$= \iint_D \vec{F} \cdot \frac{(-g_x \hat{i} - g_y \hat{j} + \hat{k})}{\|\nabla f\|} \|\nabla f\| \, dA$$

$$= \iint_D \vec{F} \cdot \left(-g_x \hat{i} - g_y \hat{j} + \hat{k}\right) dx\,dy$$

- Similarly for $x = h(y, z)$ and $y = \psi(x, z)$.

- Parameteric Form

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} \, ds$$

$$ds = \|\vec{r}_u \times \vec{r}_v\| \, du\,dv$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iint_D \vec{F}\cdot\left(\vec{r}_u \times \vec{r}_v\right) du\, dv$$

Example 1 : Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$

$\vec{F} = z^2\hat{i} + xy\hat{j} - y^2\hat{k}$

S : Portion of the surface of the cylinder $x^2+y^2=36$, $0 \le z \le 4$ included in the first octant.

Solution: S: $x^2+y^2=36$, $0\le z\le 4$

$\vec{r}(\theta, z) = 6\cos\theta\,\hat{i} + 6\sin\theta\,\hat{j} + z\hat{k}$

$\vec{r}_\theta = -6\sin\theta\,\hat{i} + 6\cos\theta\,\hat{j} + \hat{k}$

$\vec{r}_z = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6\sin\theta & 6\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\|\vec{r}_\theta \times \vec{r}_z\| = 6$$

$$\vec{F} \cdot (\vec{r}_\theta \times \vec{r}_z) = 6z^2 \cos\theta + 6xy \sin\theta$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_D 6z^2 \cos\theta + 6xy \sin\theta \, d\theta\, dz$$

$$= \int_{z=0}^{4} \int_{\theta=0}^{\pi/2} 6z^2 \cos\theta + 216 \sin^2\theta \cos\theta \, d\theta\, dz$$

$$= 416$$

Stoke's Theorem: Let S be an orientable smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also, let $\vec{F}$ be a vector field, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

Example: Use stoke's theorem to evaluate $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$

$\vec{F} = z^2\,\hat{i} + y^2\,\hat{j} + x\,\hat{k}$

C: Triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Solution: $x+y+z=1$

$$z = 1-x-y = g(x,y)$$

$$f(x,y,z) = z-1+x+y$$

$$\nabla f = \hat{i}+\hat{j}+\hat{k}$$

$$\|\nabla f\| = \sqrt{3}$$

$$\hat{n} = \frac{\nabla f}{\|\nabla f\|} \quad ; \quad \text{curl}\,\vec{F} = (2z-1)\,\hat{j}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{s}$$

$$= \iint_D (2z-1)\,\hat{j}\cdot(\hat{i}+\hat{j}+\hat{k})\,dA$$

$$= \int_0^1 \int_0^{1-x} \left(2(1-x-y)-1\right) dy\,dx$$

$$= \int_0^1 \left(y-2xy-y^2\right)_0^{1-x} dx$$

$$= \frac{-1}{6}$$

Divergence Theorem: Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives. Then

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_E \operatorname{div} \vec{F} \, dV$$

Example: Evaluate $\iint_S \vec{F} \cdot d\vec{s}$

$\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^3\hat{k}$ taken over the region bounded by the cylinder

$x^2 + y^2 = 4$, $z = 0$ and $z = 3$

Solution:

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_E \operatorname{div}\vec{F}\, dv$$

$$\operatorname{div}\vec{F} = 4 - 4y + 3z$$

$$= \int_{z=0}^{3} \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+3z)\, dy\, dx\, dz$$

$$= 84\pi$$

Example: Verify Stoke's theorem for the vector field

$$\vec{V} = (3x-y)\,\hat{i} - 2yz^2\,\hat{j} - 2y^2z\,\hat{k}$$

S: surface of the sphere $x^2+y^2+z^2=16$, $z>0$.

Proof: Stoke's theorem

$$\int_C \vec{V}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{V}\cdot d\vec{s}$$

L.H.S

$$\int_C \vec{V}\cdot d\vec{r} = \oint_C (3x-y)\,dx - 2yz^2\,dy - 2y^2z\,dz \quad -(1)$$

(Let s be the projection on the xy plane, $x^2+y^2\leq 16$, $z=0$ (Projection is circular region)

C: boundary of the circle

$$x^2+y^2=16,\quad z=0$$

(1) becomes

$$\int_C \vec{V}\cdot d\vec{r} = \oint_C (3x-y)\,dx$$

$x = 4\cos\theta$

$y = 4\sin\theta$

$dx = -4\sin\theta \, d\theta$

$$\oint_C (3x-y)\,dx = \int_0^{2\pi} (3\cdot 4\cos\theta - 4\sin\theta)(-4\sin\theta)\,d\theta$$

$$= 16\pi$$

R.H.S :

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$= \hat{k}$$

$$\vec{n} = \nabla f(x,y,z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{4}$$

$$\nabla \times \vec{v} \cdot \hat{n} = \frac{z}{4}$$

$$\iint_S \nabla \times \vec{v} \cdot \hat{n} \, ds = \iint_D \frac{z}{4}\,ds = \iint_D \frac{z}{4} \frac{dx\,dy}{z/4}$$

$$= \int_0^4 \int_0^{2\pi} r\,dr\,d\theta = 16\pi$$