

Gamma Function: The improper Integral defined by

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx ; n > 0$$

is called Gamma function.

Properties of Gamma function:

$$1. \quad \Gamma(1) = \int_0^{\infty} e^{-x} dx = 1$$

$$\begin{aligned} 2. \quad \cancel{\Gamma(x) \neq 1} \quad \Gamma(n+1) &= \int_0^{\infty} e^{-x} x^n dx \\ &= \left[-x^n e^{-x} \right]_0^{\infty} + n \int_0^{\infty} x^{n-1} e^{-x} dx \\ &= n \int_0^{\infty} e^{-x} x^{n-1} dx \\ &= n \Gamma(n) \end{aligned}$$

Thus $\Gamma(n+1) = n \Gamma(n)$

3. $\sqrt{n+1} = n!$ for any positive integer n .

Proof:

$$\begin{aligned}\sqrt{n+1} &= n \sqrt{n} \\ &= n(n-1) \sqrt{n-1} \\ &\vdots \\ &= n(n-1)(n-2) \dots 1 \cdot \sqrt{1} \\ &= n!\end{aligned}$$

4. $\sqrt{n+1} = n!$

Take $n = 0$

$$\sqrt{1} = 0! = 1$$

Beta Function: The improper

integral defined by

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$m > 0, n > 0$

is called the Beta function.

Exc: 1. $\int_0^1 x^{1/2} (1-x)^4 dx = \int_0^1 x^{3/2-1} (1-x)^{5-1} dx$

$= \beta(3/2, 5)$

2. $\int_0^1 x^{1/2} (1-x)^{-3} dx$ is not a Beta function

as $m = 3/2 > 0$

$n = -2 < 0$

Properties of Beta Function

$$1. \quad \beta(m, n) = \beta(n, m)$$

Solution:

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$\therefore \beta(m, n) = \int_0^1 (1-x)^{m-1} (1-1+x)^{n-1} dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$= \beta(n, m)$$

$$\begin{aligned}
 2. \quad \beta(m, n) &= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \\
 &= 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cos^{2m-1} \theta d\theta
 \end{aligned}$$

Proof:

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Put } x = \sin^2 \theta d\theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2} \theta \cos^{2n-2} \theta \cdot 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$3. \quad \beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\underline{\text{Proof:}} \quad \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\text{Take } x = \frac{t}{1+t} \Rightarrow t = \frac{x}{1-x}$$

$$dx = \frac{dt}{(1+t)^2}$$

$$\beta(m, n) = \int_0^{\infty} \left(\frac{t}{1+t} \right)^{m-1} \left(\frac{1}{1+t} \right)^{n-1} \frac{dt}{(1+t)^2}$$

$$= \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt$$

$$= \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Also

$$\beta(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad \left(\begin{array}{l} \text{As } \beta(m, n) \\ = \beta(n, m) \end{array} \right)$$

$$4. \quad \beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx,$$

Proof: $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$ $m > 0, n > 0$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= I_1 + I_2 \quad - \textcircled{1}$$

$$I_2 = \int_1^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$

$$= \int_1^0 \frac{(1/t)^{n-1}}{\left(1 + \frac{1}{t}\right)^{m+n}} \left(-\frac{1}{t^2}\right) dt$$

$\left(\because x = \frac{1}{t} \rightarrow 1 \Rightarrow t = 1 \right.$
 $\left. \rightarrow \infty \Rightarrow t = 0 \right)$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow \beta(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx; \quad \begin{matrix} m > 0 \\ n > 0 \end{matrix}$$

Exc: Show that

$$\int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = 2\beta(m, n)$$

Solution: $\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$ — (1)

Also $\beta(m, n) = \beta(n, m)$

$$\Rightarrow \beta(n, m) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$
 — (2)

Add (1) + (2)

$$2\beta(m, n) = \int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Exc: Show that

$$\int_0^a x^{m-1} (a-x)^{n-1} dx = a^{m+n-1} \beta(m, n)$$

Solution: Put $x = ay$
 $dx = a dy$

L.H.S

$$\int_0^a x^{m-1} (a-x)^{n-1} dx$$

$$= \int_0^1 (ay)^{m-1} a^{n-1} (1-y)^{n-1} a dy$$

$$= a^{m+n-1} \int_0^1 y^{m-1} (1-y)^{n-1} dy$$

$$= a^{m+n-1} \beta(m, n)$$

Exc: Prove that

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$m > -1, n > -1$

Solution:

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \int_0^{\pi/2} \sin^{m-1} \theta \cos^{n-1} \theta \sin \theta \cos \theta d\theta$$

$$= \int_0^{\pi/2} \sin^{m-1} \theta (1 - \sin^2 \theta)^{\frac{n-1}{2}} \sin \theta \cos \theta d\theta$$

Put ~~x~~ $\sin^2 \theta = x$

$$2 \sin \theta \cos \theta d\theta = dx$$

$$= \int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} \frac{dx}{2}$$

$$= \frac{1}{2} \int_0^1 x^{\frac{m+1}{2} - 1} (1-x)^{\frac{n+1}{2} - 1} dx$$

$$= \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Lemma:
$$\int_0^{\infty} y^{n-1} e^{-1y} dy = \frac{\Gamma n}{1^n}$$

Proof:
$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Put $x = 1y$

$dx = 1 dy$

$$\therefore \Gamma n = \int_0^{\infty} e^{-1y} (1y)^{n-1} 1 dy$$

$$= 1^n \int_0^{\infty} y^{n-1} e^{-1y} dy$$

$$\therefore \int_0^{\infty} y^{n-1} e^{-1y} dy = \frac{\Gamma n}{1^n}$$

Prove that $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

Proof: We know

$$\int_0^{\infty} e^{-xy} y^{n-1} dy = \frac{\Gamma(n)}{x^n} \quad \left(\begin{array}{l} 1=x \\ \text{above Lemma} \end{array} \right)$$

or $\Gamma(n) = \int_0^{\infty} x^n e^{-xy} y^{n-1} dy \quad \text{--- (1)}$

Also $\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x} dx \quad \text{--- (2)}$

Multiplying both sides of eq (1) by

$$x^{m-1} e^{-x}$$

$$\Gamma(n) x^{m-1} e^{-x} = \int_0^{\infty} x^{m+n-1} y^{n-1} e^{-x(y+1)} dy \quad \text{--- (3)}$$

Integrate (3) w.r.t x from

$$x=0 \quad \text{to} \quad x=\infty.$$

$$= \int_0^{\infty} y^{n-1} \left(\int_0^{\infty} x^{m+n-1} e^{-(y+1)x} dx \right) dy$$

But
$$\int_0^{\infty} x^{m+n-1} e^{-(y+1)x} dx = \frac{\Gamma(m+n)}{(1+y)^{m+n}}$$

$\therefore$ equation (4) becomes

$$\overline{|n|} \overline{|m|} = \int_0^\infty \overline{|m+n|} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \frac{1}{m+n} \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \lceil m+n \rceil \beta(m, n)$$

$$\Rightarrow \beta(m, n) = \sqrt{m} \sqrt{n} / \sqrt{m+n}$$

Show that $\Gamma(1/2) = \sqrt{\pi}$

Solution: As $\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$

Put $m = n = 1/2$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$\Rightarrow \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{1/2-1} (1-x)^{1/2-1} dx$$

$$= \int_0^1 \frac{dx}{\sqrt{x} \sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}$$

$$= \left[\sin^{-1} \left[\frac{x - \frac{1}{2}}{1/2} \right] \right]_0^1$$

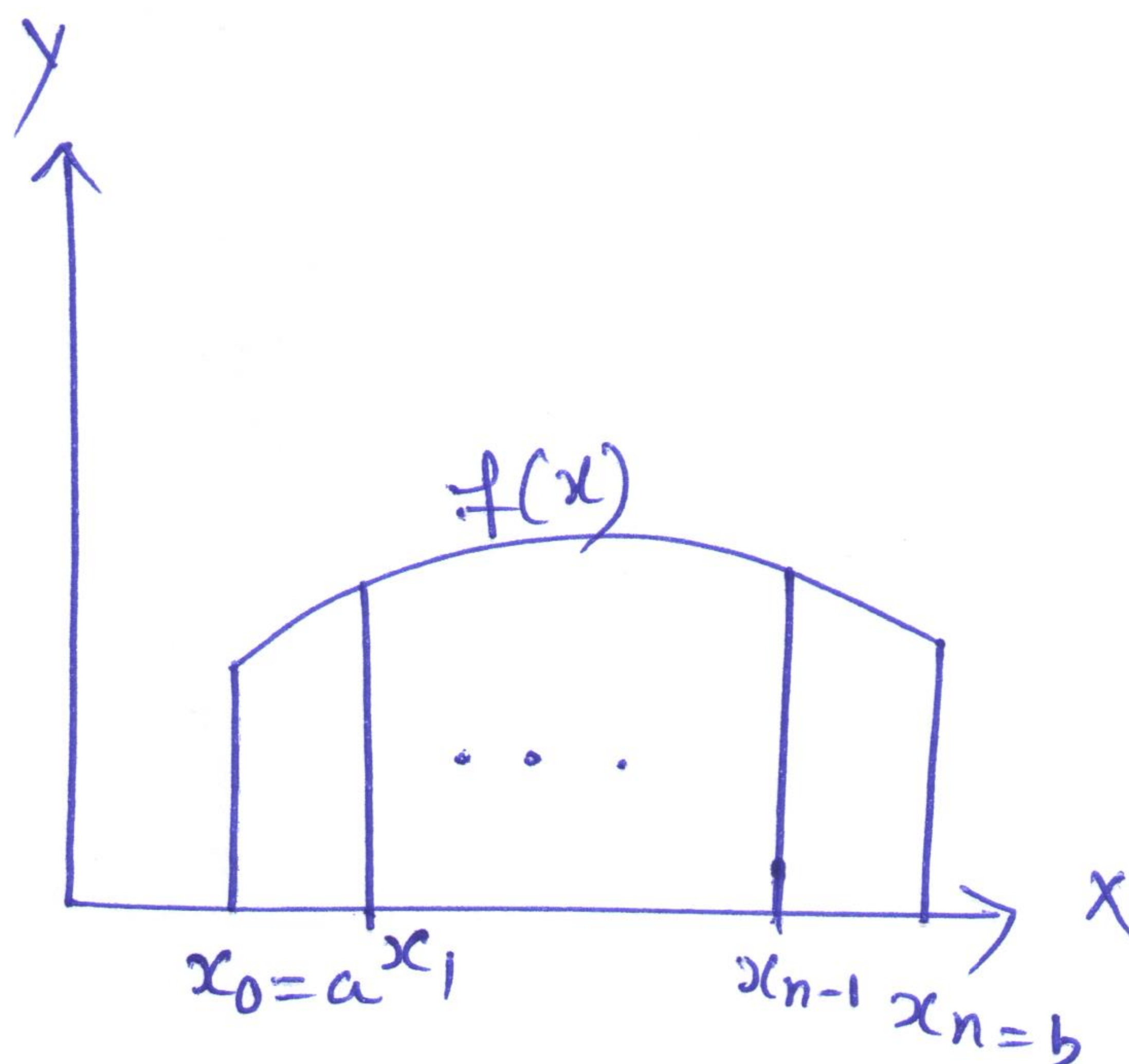
$$= \sin^{-1} 1 - \sin^{-1} (-1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

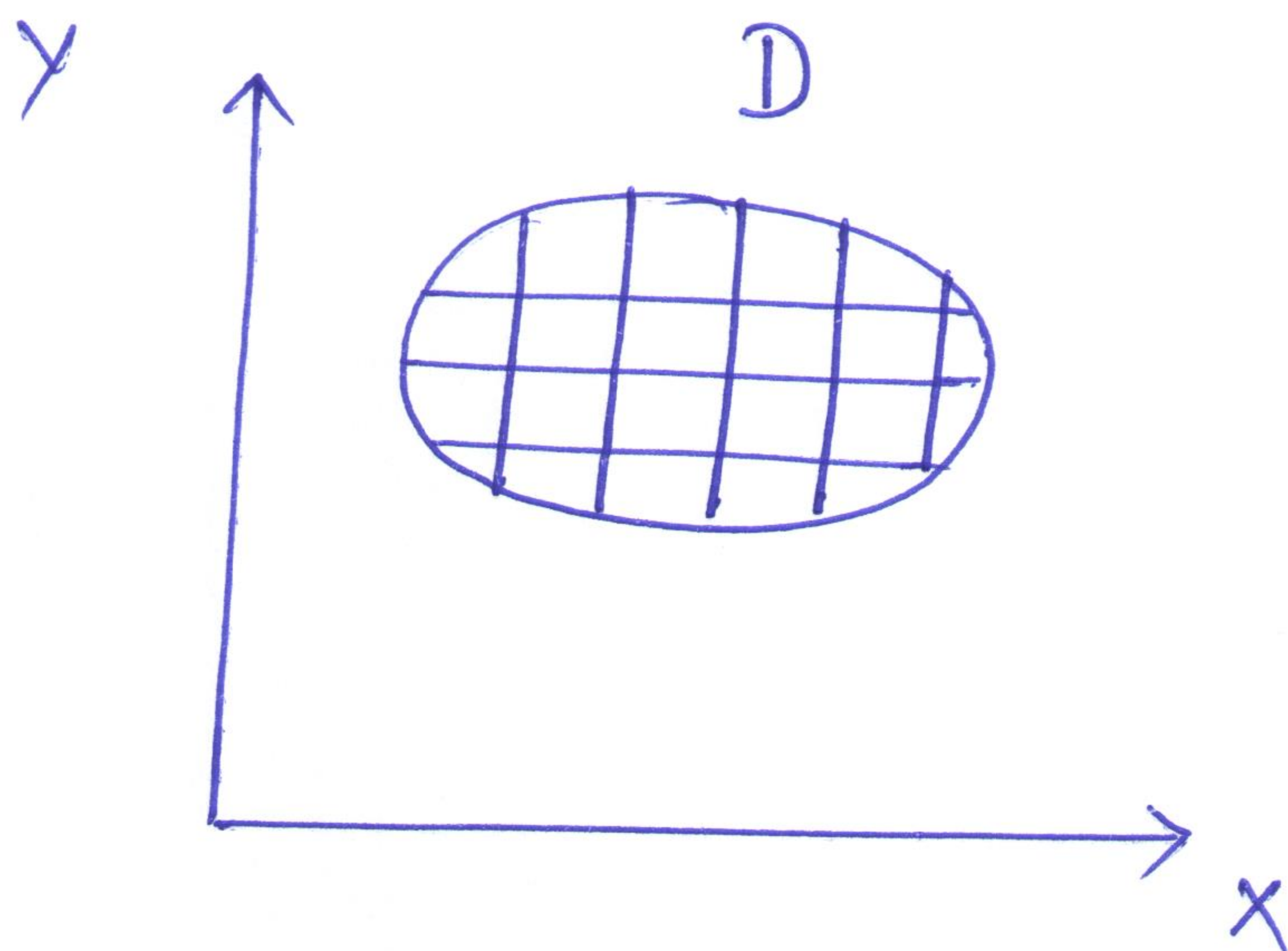
$$\Rightarrow \sqrt{1/2} = \sqrt{\pi}$$

Integrals of function of single variable:



$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k$$

Double Integrals

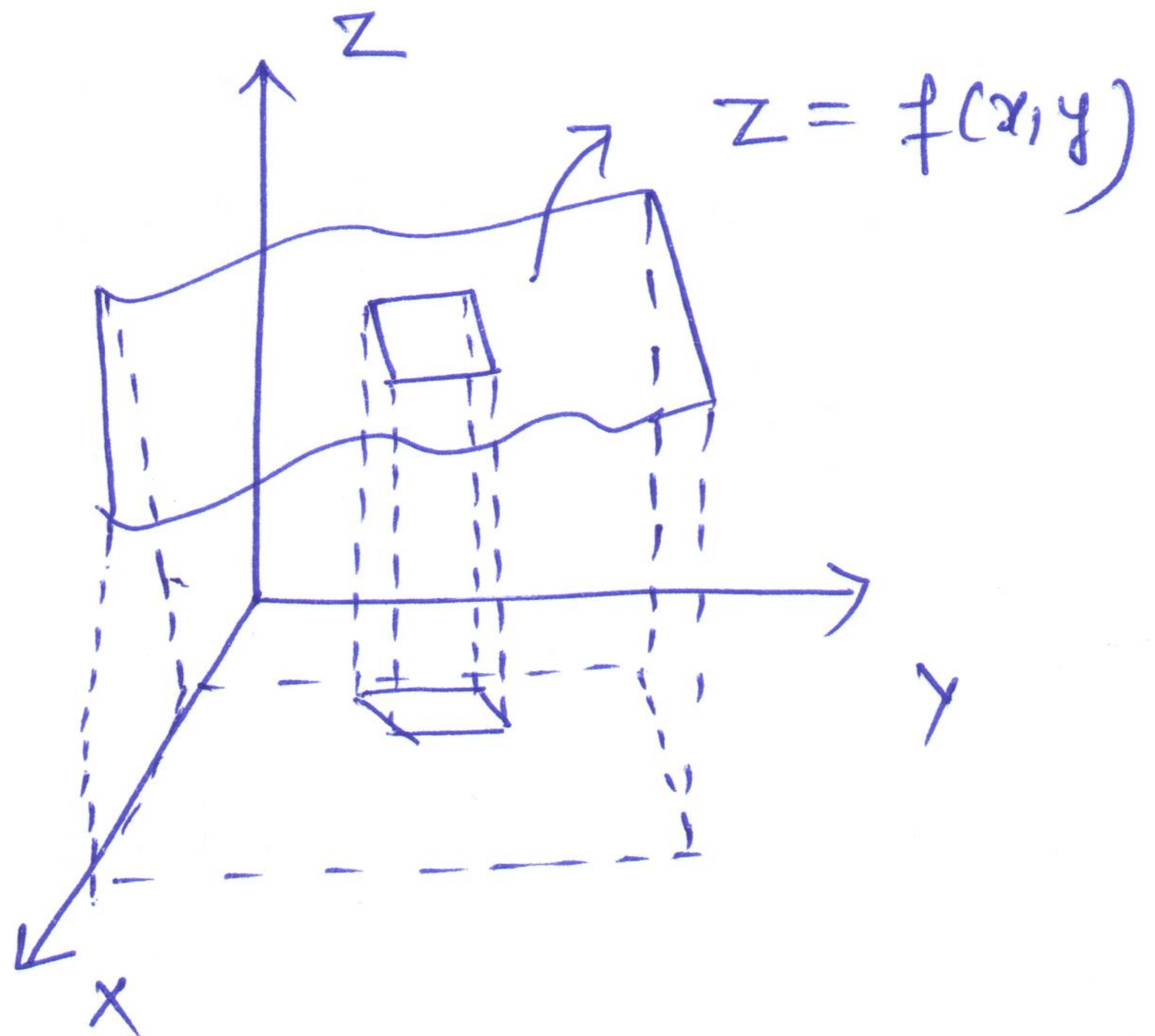


- Let $f(x, y)$ be defined in a closed region D of the xy -plane.
- Divide D into n subregions of area ΔA_j , $j = 1, 2, \dots, n$.
- Let (x_j, y_j) be some point of ΔA_j
- Then consider
$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j, y_j) \Delta A_j$$
- If this limit exists, then it is denoted by

$$\iint_D f(x, y) dA \quad \text{or} \quad \iint_D f(x, y) dx dy \quad \text{or} \quad \iint_D f(x, y) dy dx$$

Note: Above Limit exists if
 $z = f(x, y)$ is Continuous
 or piecewise continuous in D .

Geometrical Interpretation



$$\bullet \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j, y_j) \Delta x \Delta y$$

$$= \iint_D f(x, y) dx dy \text{ represents volume}$$

OR

Area of D if $f(x, y) = 1$

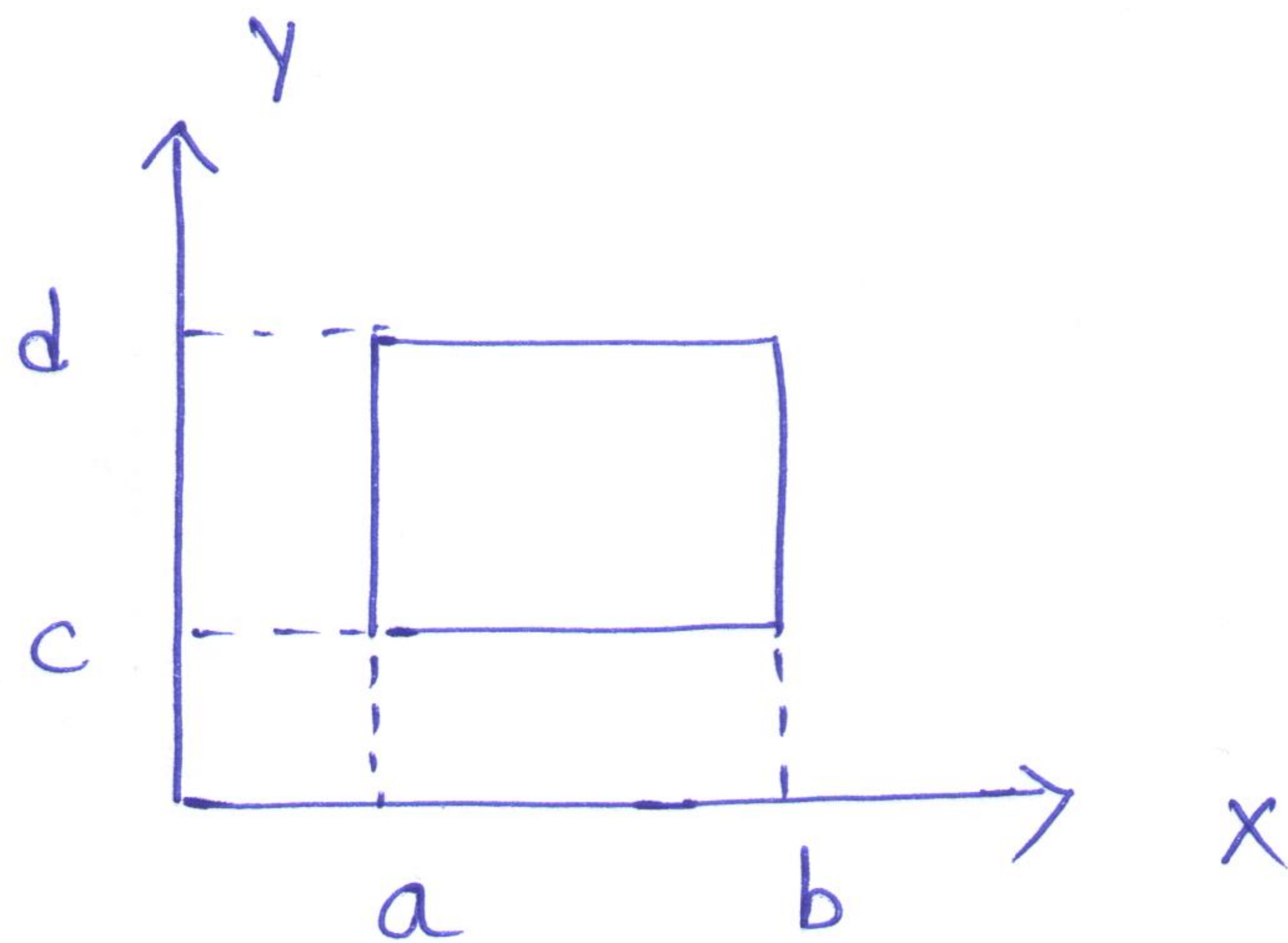
Properties

- $\iint_D k f(x, y) dA = k \iint_D f(x, y) dA$
- $\iint_D [f(x, y) \pm g(x, y)] dA$
 $= \iint_D f(x, y) dA \pm \iint_D g(x, y) dA$
- $\iint_D f(x, y) dA \geq 0$ if $f(x, y) \geq 0$ on D
- $\iint_D f(x, y) dA \geq \iint_D g(x, y) dA$
 if $f(x, y) \geq g(x, y)$ on D .
- $\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$
 if $D = D_1 \cup D_2$

Evaluation of double Integral

- If $f(x, y)$ is continuous
(or defined and bounded) on
rectangular region

$$D: a \leq x \leq b, \quad c \leq y \leq d$$

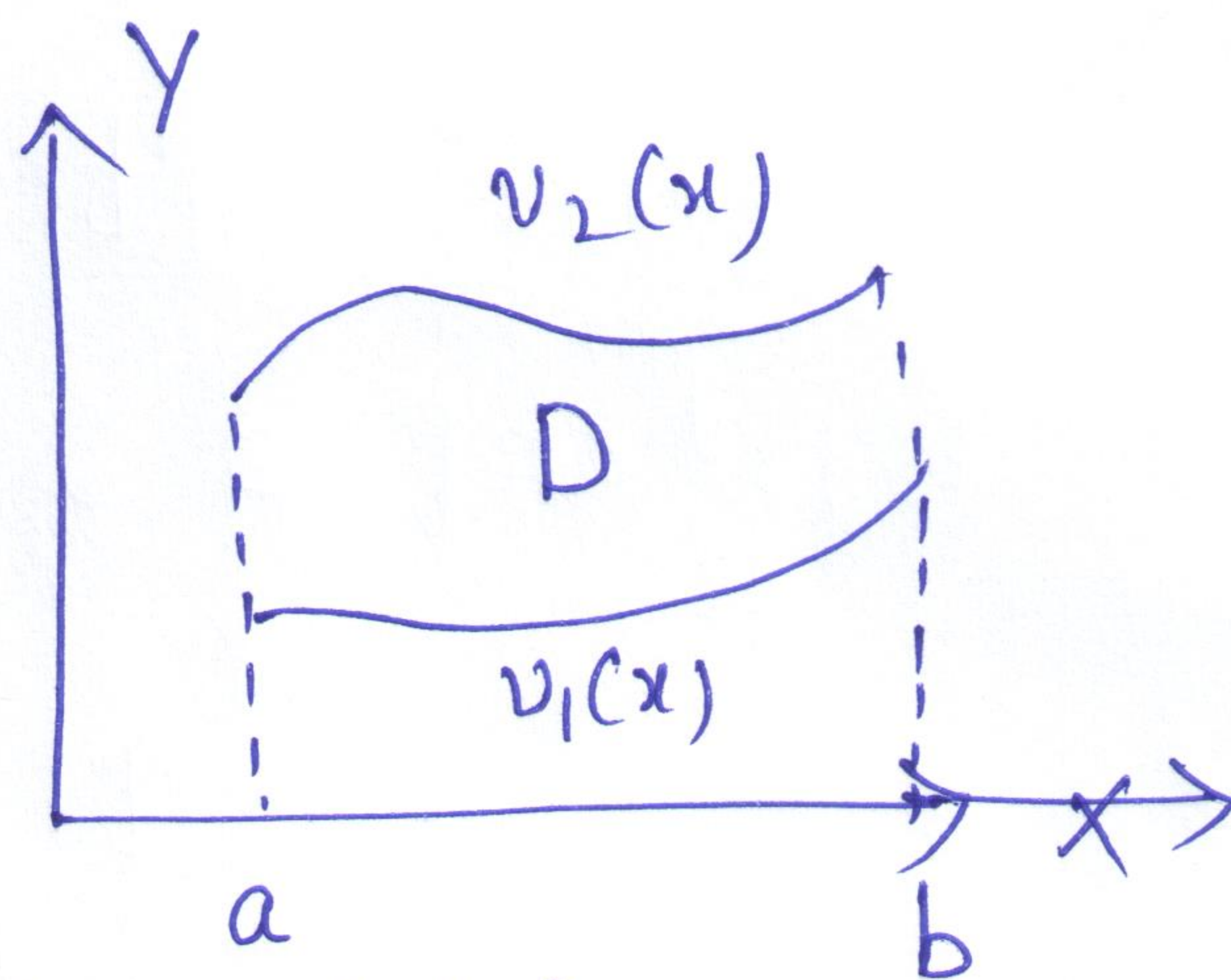


$$\begin{aligned} \iint_D f(x, y) dA &= \int_c^d \int_a^b f(x, y) dx dy \\ &\quad \underbrace{\int_a^b f(x, y) dx}_{\phi(y)} \\ &= \int_a^b \int_c^d f(x, y) dy dx \\ &\quad \underbrace{\int_c^d f(x, y) dy}_{\phi(x)} \end{aligned}$$

Evaluation of double Integral

(203)

- If $f(x, y)$ is defined and bounded in D
- $v_1(x)$ and $v_2(x)$ are continuous in (a, b) .

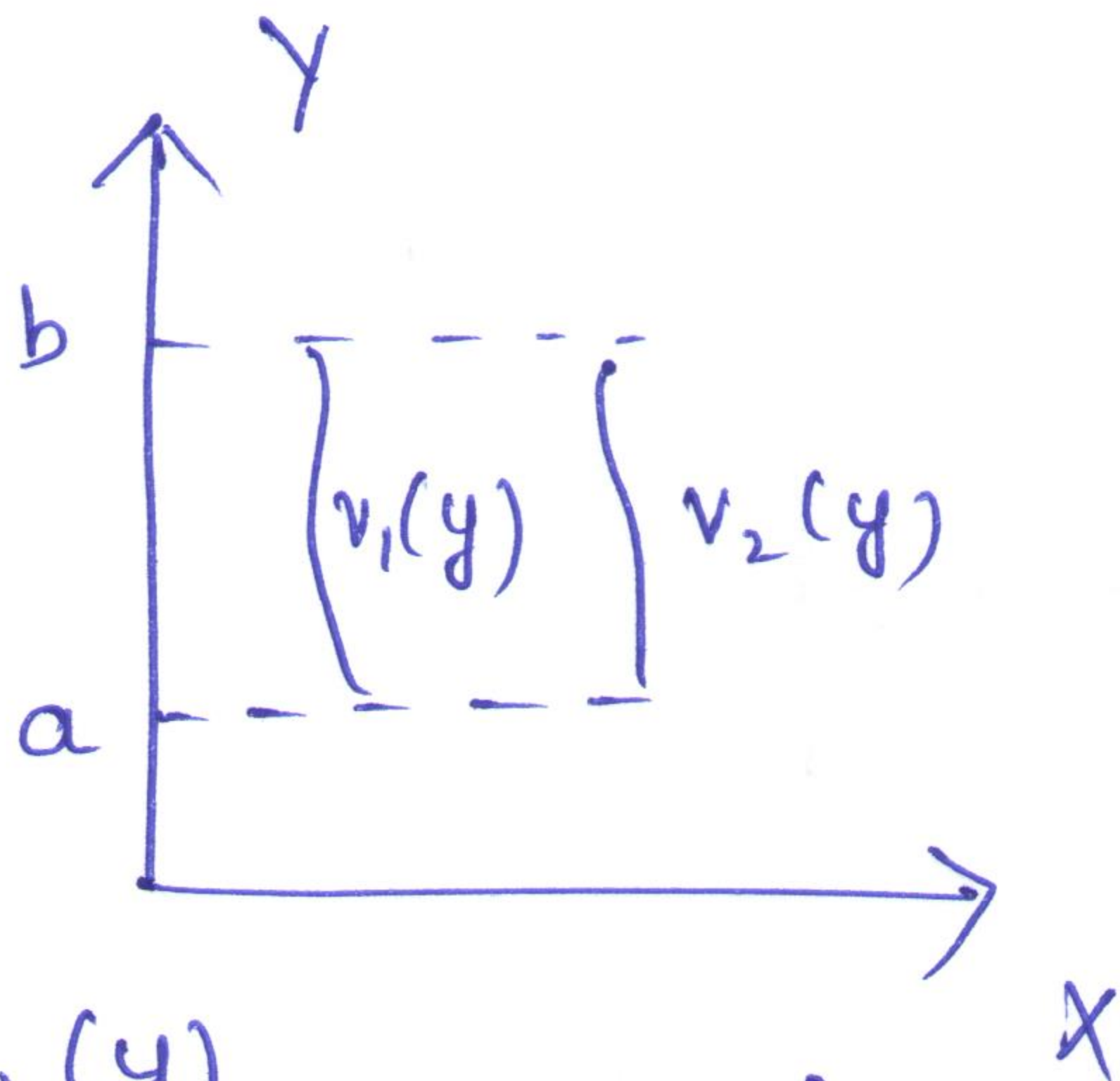


$$\iint_D f(x, y) dA = \int_a^b \left(\int_{v_1(x)}^{v_2(x)} f(x, y) dy \right) dx$$

(Non Rectangular region)

Evaluation of double Integral

- If $f(x, y)$ is defined and bounded in D .
- $v_1(y)$ and $v_2(y)$ are continuous in (a, b)

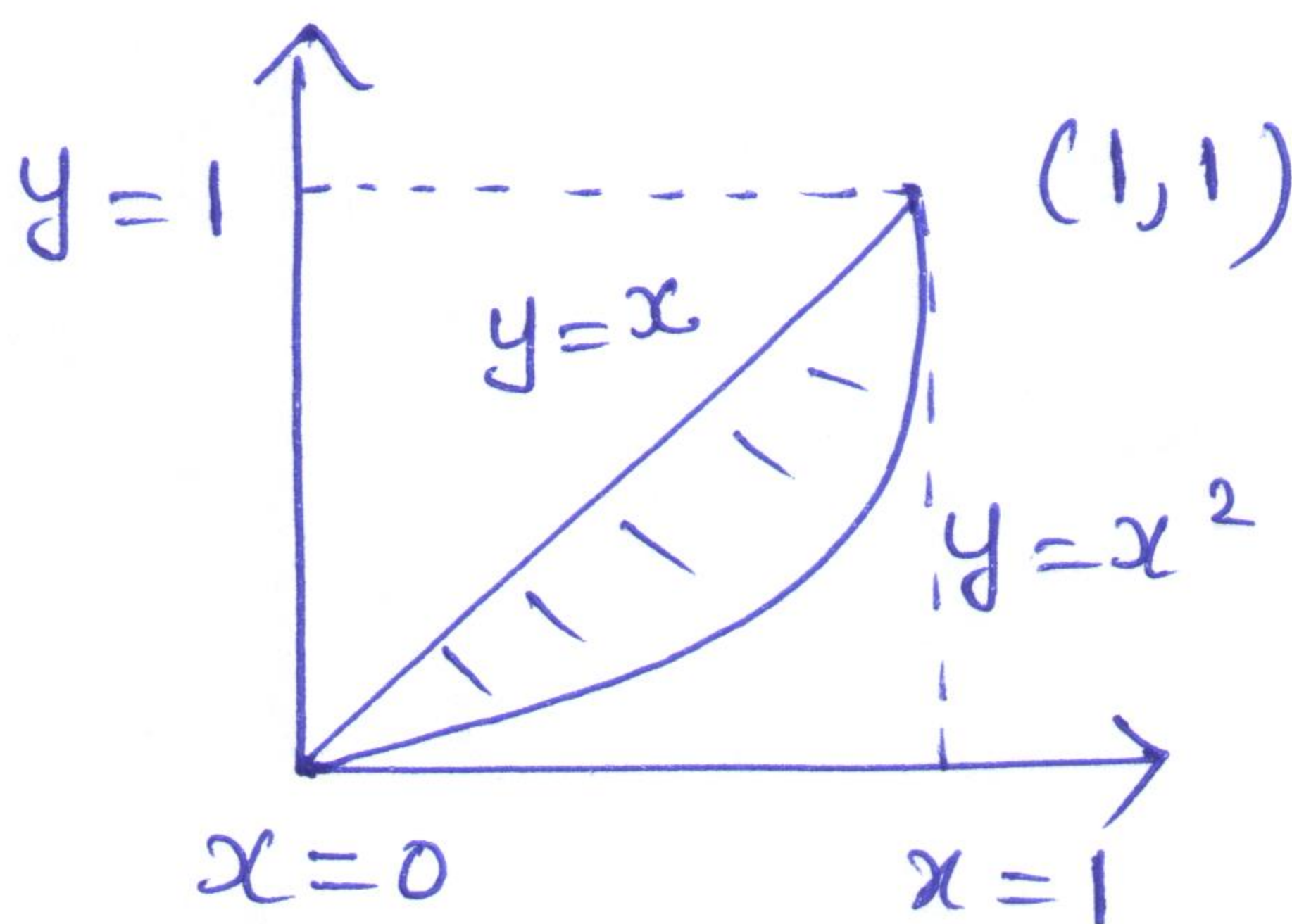


$$\iint_D f(x, y) dA = \int_a^b \left(\int_{v_1(y)}^{v_2(y)} f(x, y) dx \right) dy$$

Example 1:

$$\iint_R xy(x+y) dA$$

Where R is the region bounded by the line $y=x$ and the curve $y=x^2$.



$$\begin{aligned} \iint_R xy(x+y) dA &= \int_{x=0}^1 \int_{y=x^2}^{y=x} xy(x+y) dy dx \\ &= \int_{y=0}^1 \int_{x=y}^{\sqrt{y}} xy(x+y) dx dy \end{aligned}$$

Consider $\int_{x=0}^1 \int_{x^2}^x xy(x+y) dy dx$

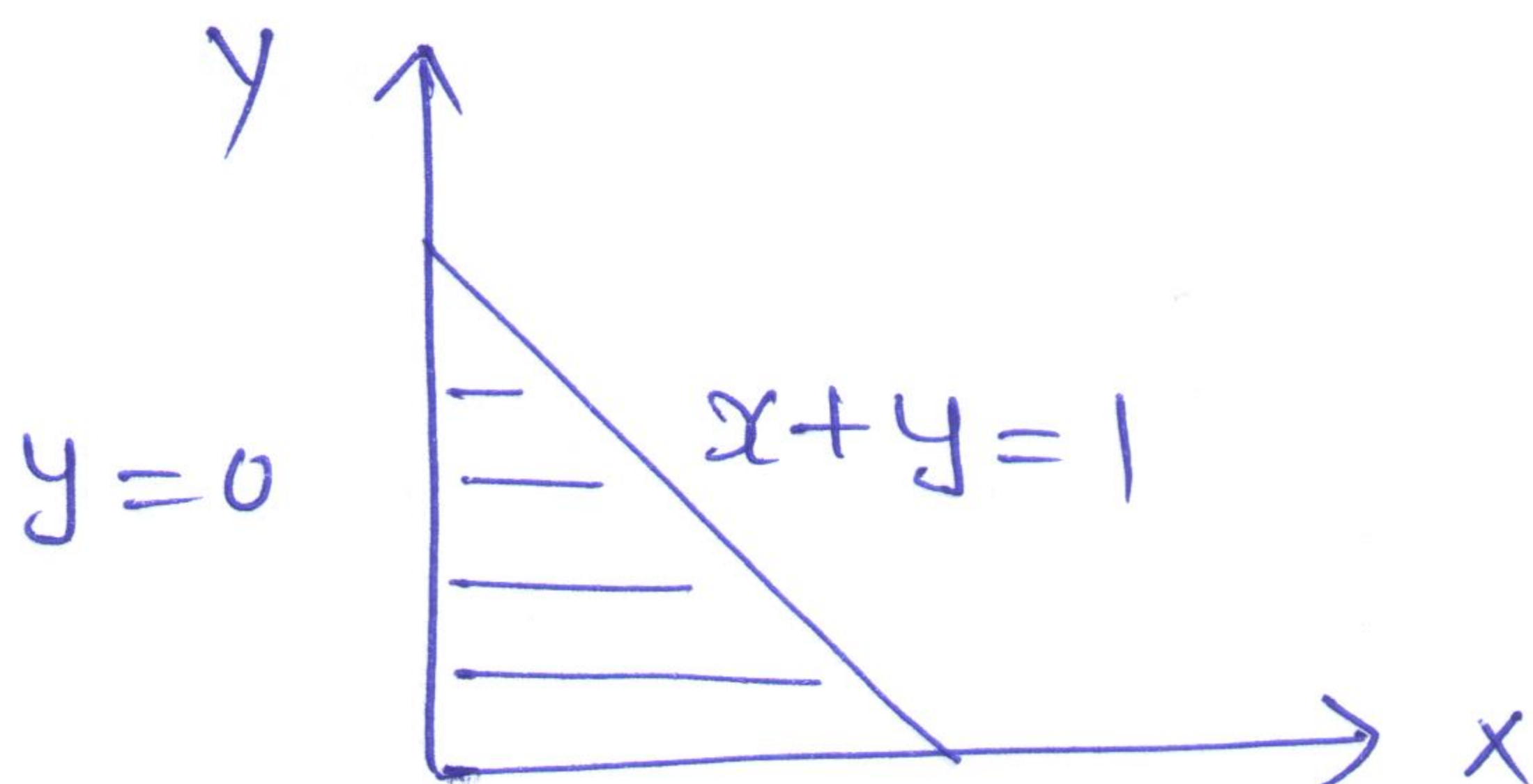
$$= \int_0^1 \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} dx$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{\cancel{3}}{\cancel{36}} - \frac{3}{56}$$

Exc: $\iint_R e^{2x+3y} dx dy$

Where R is the region bounded by $x=0$, $y=0$ and $x+y=1$.



$$\begin{aligned} \iint_R e^{2x+3y} dx dy &= \int_{x=0}^1 \int_{y=0}^{1-x} e^{2x+3y} dy dx \\ &= \int_{y=0}^1 \int_{x=0}^{1-y} e^{2x+3y} dx dy \end{aligned}$$

Consider

$$\begin{aligned} \int_{x=0}^1 \int_0^{1-x} e^{2x+3y} dy dx &= \frac{1}{3} \int_0^1 e^{2x} (e^{3-3x} - 1) dx \\ &= \frac{1}{3} \left[-\frac{3e^2}{2} + e^3 + \frac{1}{2} \right] \end{aligned}$$

Ex. Show that

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dx \right] dy \neq \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dy \right] dx.$$

L.H.S.

$$\begin{aligned} \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx \right) dy &= \int_0^1 \left(\int_0^1 \frac{x+y-2y}{(x+y)^3} dx \right) dy \\ &= \int_0^1 \left[\int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right) dx \right] dy = \int_0^1 \left[\frac{1}{1+y} + \frac{2y}{2(1+y)^2} \right] dy \\ &= \int_0^1 \frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y} \right]_0^1 = -\frac{1}{2} \end{aligned}$$

Similarly, we can show R.H.S

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dy \right] dx = \frac{1}{2}.$$

Hence, L.H.S $\neq$ R.H.S.

The reason is that the function $f = \frac{x-y}{(x+y)^3}$ is not cts in $R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Note: Both exist but not equal.

Ex 1:

$$\int_1^{\log 8} \int_{x=0}^{\log y} e^{x+y} \underline{dx} \underline{dy}$$

$$I = \int_1^{\log 8} \left(\int_0^{\log y} e^x dx \right) e^y dy = \int_1^{\log 8} [e^x]_0^{\log y} e^y dy.$$

$$= \int_1^{\log 8} (e^{\log y} - 1) e^y dy = \int_1^{\log 8} e^y (y - 1) dy$$

$$= [e^y y - 2e^y]_1^{\log 8} = \boxed{8 \log 8 - 16 + e}$$

Ex 2: evaluate $\int_0^\pi \int_0^x \sin y \, dy \, dx$.

(4)

Hint: $\int_0^\pi \left(\int_0^x \sin y \, dy \right) dx = \pi$.

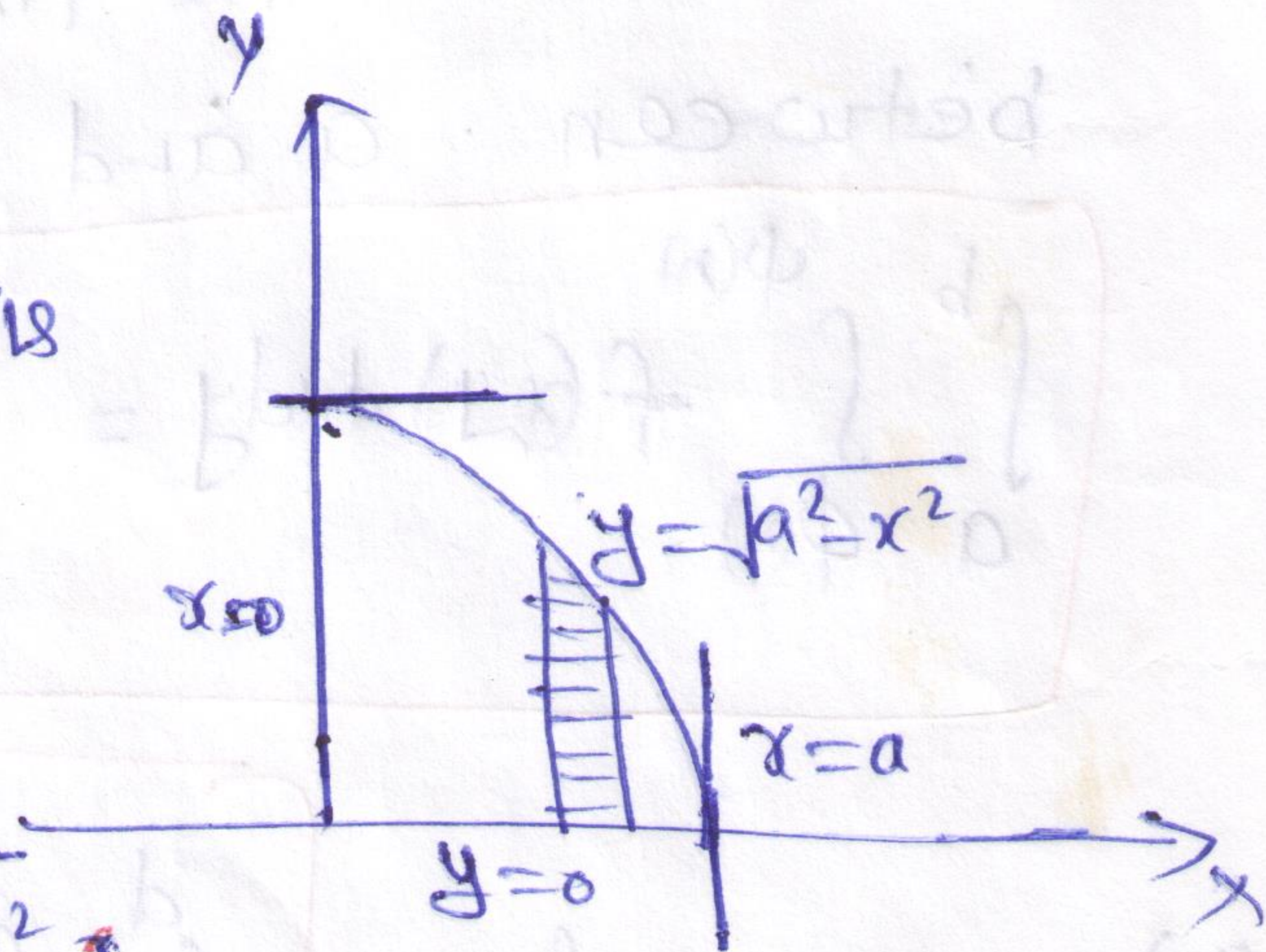
Ex 3: Evaluate $\iint_A xy \, dx \, dy$ over the +ve quadrant of the circle $x^2 + y^2 = a^2$.

Sol: Here the region of integration is

the quadrant of $x^2 + y^2 = a^2$

where x varies from "0 to a "

and y varies from 0 to $\sqrt{a^2 - x^2}$.



Hence,
$$\iint_A xy \, dx \, dy = \int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \, dx$$

$$= \int_0^a \left[\int_{y=0}^{\sqrt{a^2-x^2}} y \, dy \right] x \, dx = \int_0^a \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} x \, dx$$

$$= \frac{1}{2} \int_0^a x (a^2 - x^2) \, dx = \frac{1}{2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{1}{8} a^4.$$

Ex 4: Evaluate $\int_0^1 \int_0^{x^2} (x^2 + y^2) \, dx \, dy = \frac{26}{105}$.

Hint

$$R = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2 \}.$$

Ex. 5:

(5)

Evaluate $\iint_R xy \, dx \, dy$, R is the domain bounded by the x -axis, ordinate $x=2a$, and the curve $x^2=4ay$.

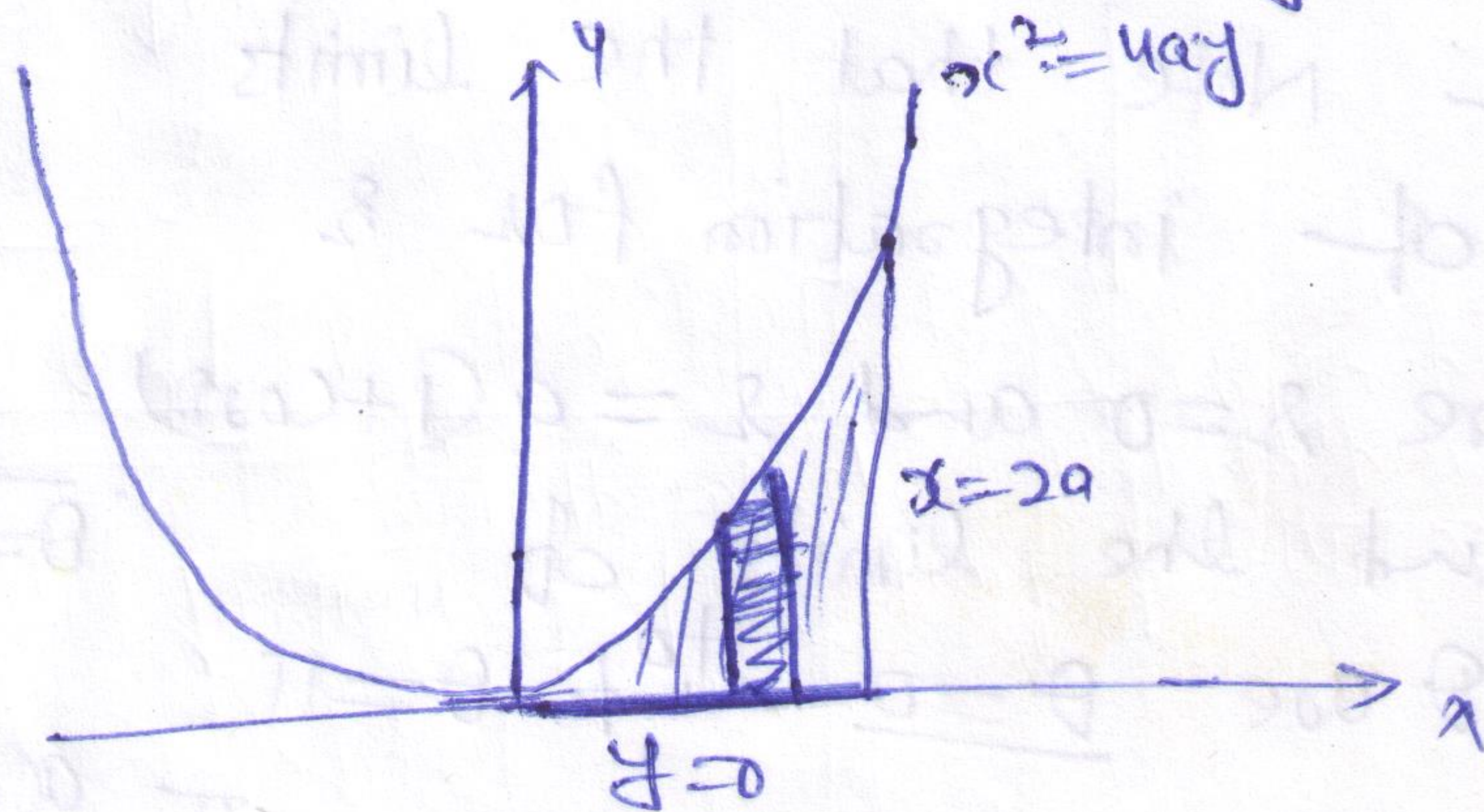
Sol: The region of integration is

$R = \{ (x,y) : 0 \leq x \leq 2a, 0 \leq y \leq \frac{x^2}{4a} \}$. The region

is bdd. by $y=0$, $x=2a$ and parabola $x^2=4ay$

$$\iint_R xy \, dx \, dy = \int_0^{2a} \int_0^{\frac{x^2}{4a}} xy \, dy \, dx$$

$$= \int_0^{2a} \left[\frac{xy^2}{2} \right]_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x^5}{32a^2} dx = \frac{a^4}{3}$$



Double Integral in Polar form

$$\iint_R f(x, y) dx dy = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta$$

- Substitute $x = r \cos \theta$, $y = r \sin \theta$
- Replace $dx dy$ by $r dr d\theta$
- G is same as R but described in Polar (form) coordinates.

Exc. Compute the area of first quadrant of a circle of radius a .

$$A = \int_{\theta=0}^{\pi/2} \int_{r=0}^a r dr d\theta = \int_0^{\pi/2} \frac{r^2}{2} \Big|_0^a d\theta$$

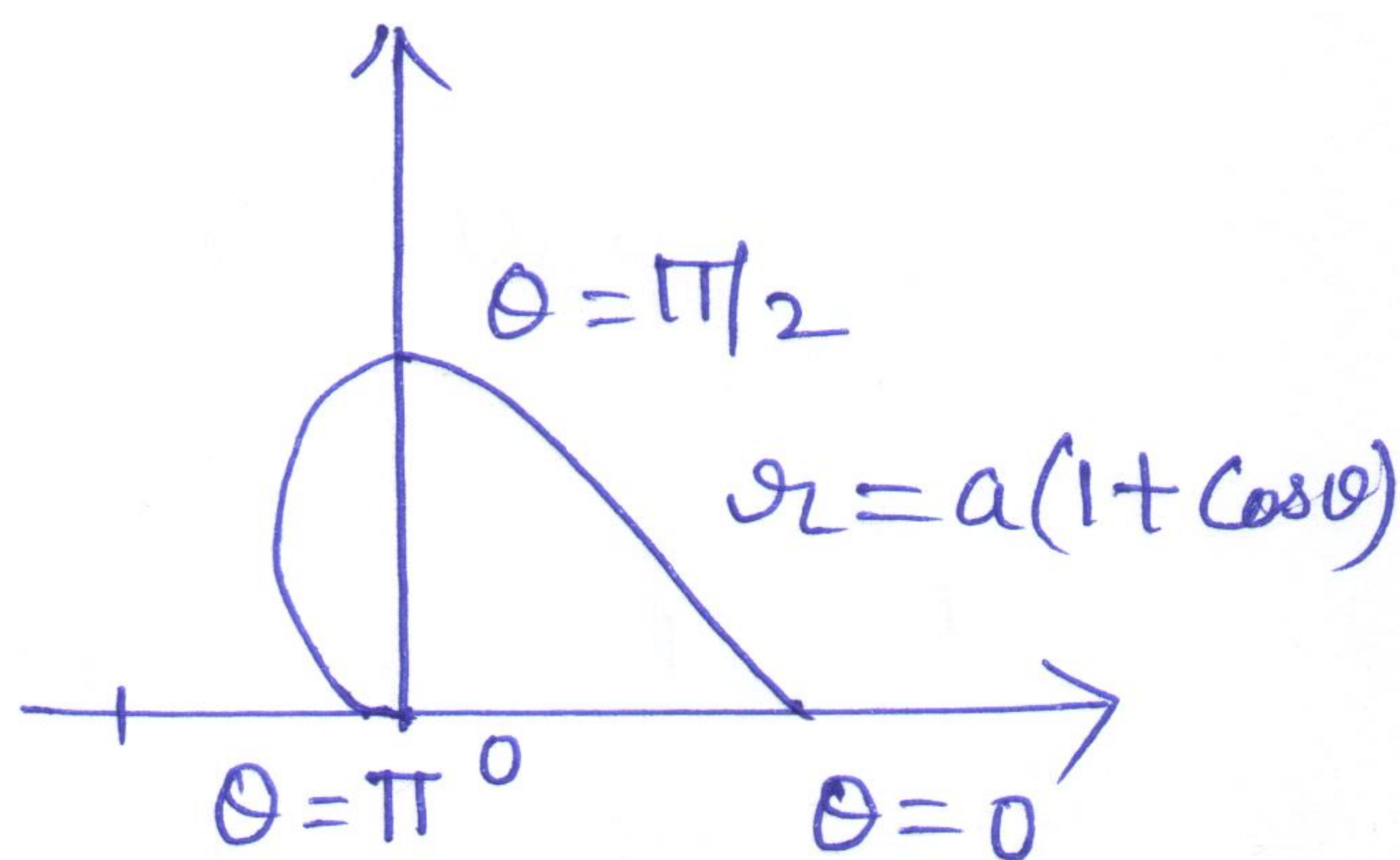
$$= \frac{\pi a^2}{4}$$

Exc:Evaluate $\iint_R r \sin \theta \, dr \, d\theta$

over the area of the cardioid

 $r = a(1 + \cos \theta)$ above the initial line.

Solution: Note the limits of the integration for $r = 0$ and $r = a(1 + \cos \theta)$ and the limits of θ are $\theta = 0$ to $\theta = \pi$.



$$\iint_R r \sin \theta \, dr \, d\theta$$

$$R \quad \pi \quad a(1 + \cos \theta)$$

$$= \int_0^\pi \int_0^{a(1 + \cos \theta)} r \sin \theta \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^\pi a^2 (1 + \cos \theta)^2 \sin \theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \left(2 \cos^2 \frac{\theta}{2} \right)^2 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \quad (210)$$

$$= 4a^2 \int_0^{\pi} \cos^5 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta$$

Put $\theta = 2\phi$

$$= 4a^2 \int_0^{\pi/2} \cos^5 \phi \sin \phi 2 d\phi$$

$$= 8a^2 \int_0^{\pi/2} \cos^5 \phi \sin \phi d\phi$$

$$= -8a^2 \left[\frac{\cos^6 \phi}{6} \right]_0^{\pi/2}$$

$$= \frac{8a^2}{6}$$

$$= \frac{4a^2}{3}$$

Ex (1) Change the order of integration in

$$I = \int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x,y) dx dy \quad \text{--- (1)}$$

Sol: The region of integration is add by

$$y = -a, \quad y = a, \quad x = 0, \quad x = \sqrt{a^2-y^2} \text{ or } x^2+y^2=a^2.$$

We have

$$I = \int_{-a}^a \left[\int_0^{\sqrt{a^2-y^2}} f(x,y) dx \right] dy.$$

Here, we first integrate w.r.t. 'x' and then w.r.t. 'y'.

On changing the order of integration,

we first integrate w.r.t 'y', along a vertical strip RS,

which extends from $y = -\sqrt{a^2 - x^2}$

to $y = \sqrt{a^2 - x^2}$.

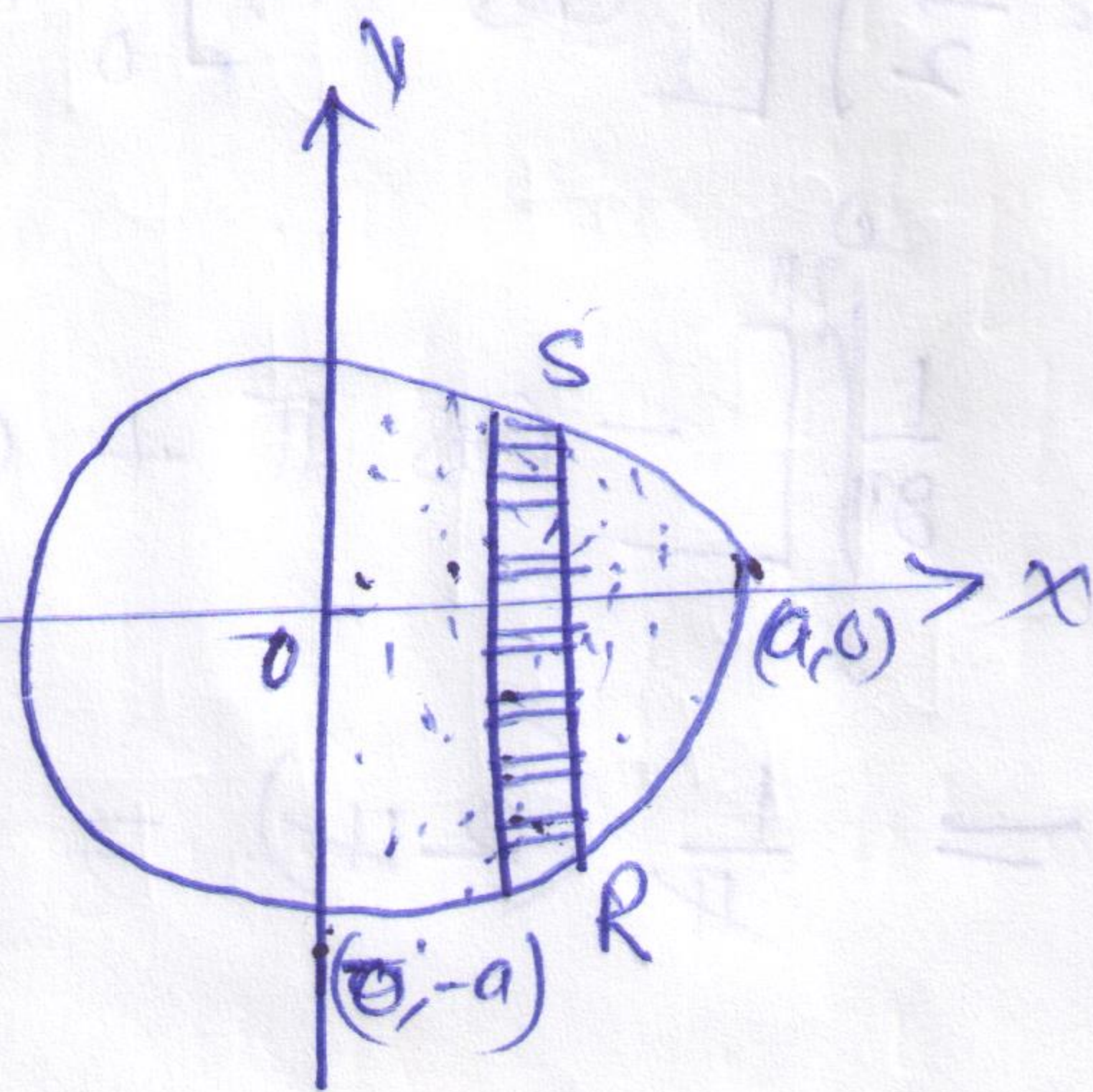
To cover the whole region

of integration, we then

integrate w.r.t x from

$x=0$ to $x=a$. Hence,

$$I = \int_0^a \left[\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y) dy \right] dx.$$



Ex. 2 Changing the order of integration, find the value of the integral

$$\int_0^{\infty} \int_x^{\infty} \frac{y-x}{y^2} dy dx.$$

Sol: The region of integration is bdd by

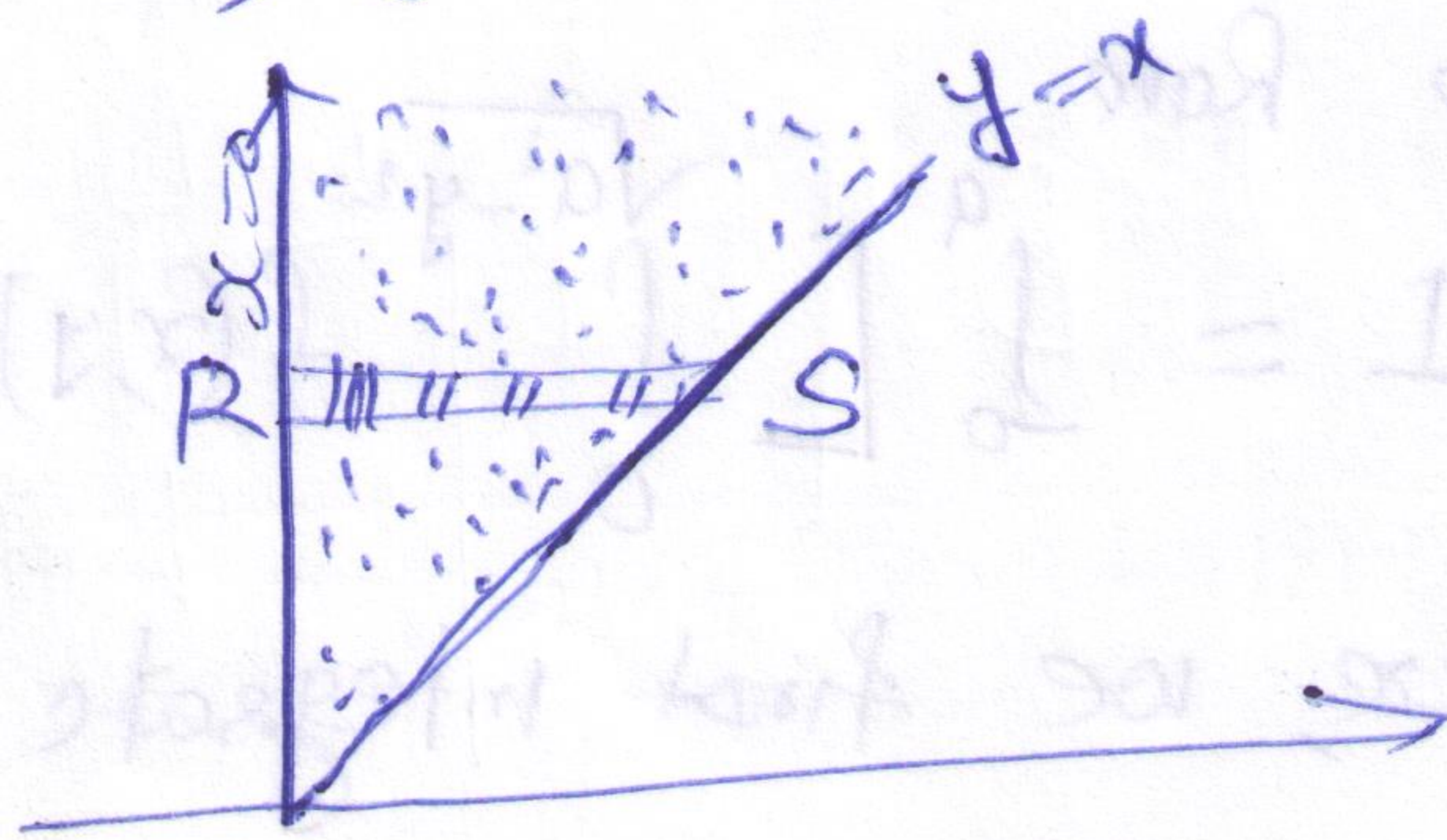
$x=0$ and $y=x$. The limits of 'x' are $\underline{x=0}$, $x=\infty$

and those of y are $y=x$, $y=\infty$. The region

is shown in the figure.

On changing the order

of integration, we



(11)

We first integrate w.r.t x , along a horizontal strip RS, which extends from $x=0$ to $x=y$.

To cover whole region of integration, we then integrate w.r.t. ' y ' from $y=0$ to ∞ .

Hence,

$$I = \int_0^{\infty} \left[\int_0^y \frac{e^{-y}}{y} dx \right] dy = \int_0^{\infty} \frac{e^{-y}}{y} [x]_0^y dy$$

$$= \int_0^{\infty} e^{-y} dy = [-e^{-y}]_0^{\infty} = -(0-1) = 1.$$

Ex. 3 Evaluate $\int_0^{\infty} \left(\int_0^x x e^{-\frac{x^2}{y}} dy \right) dx$, by changing the order of integration.

Sol: The region of integration is bdd by the lines

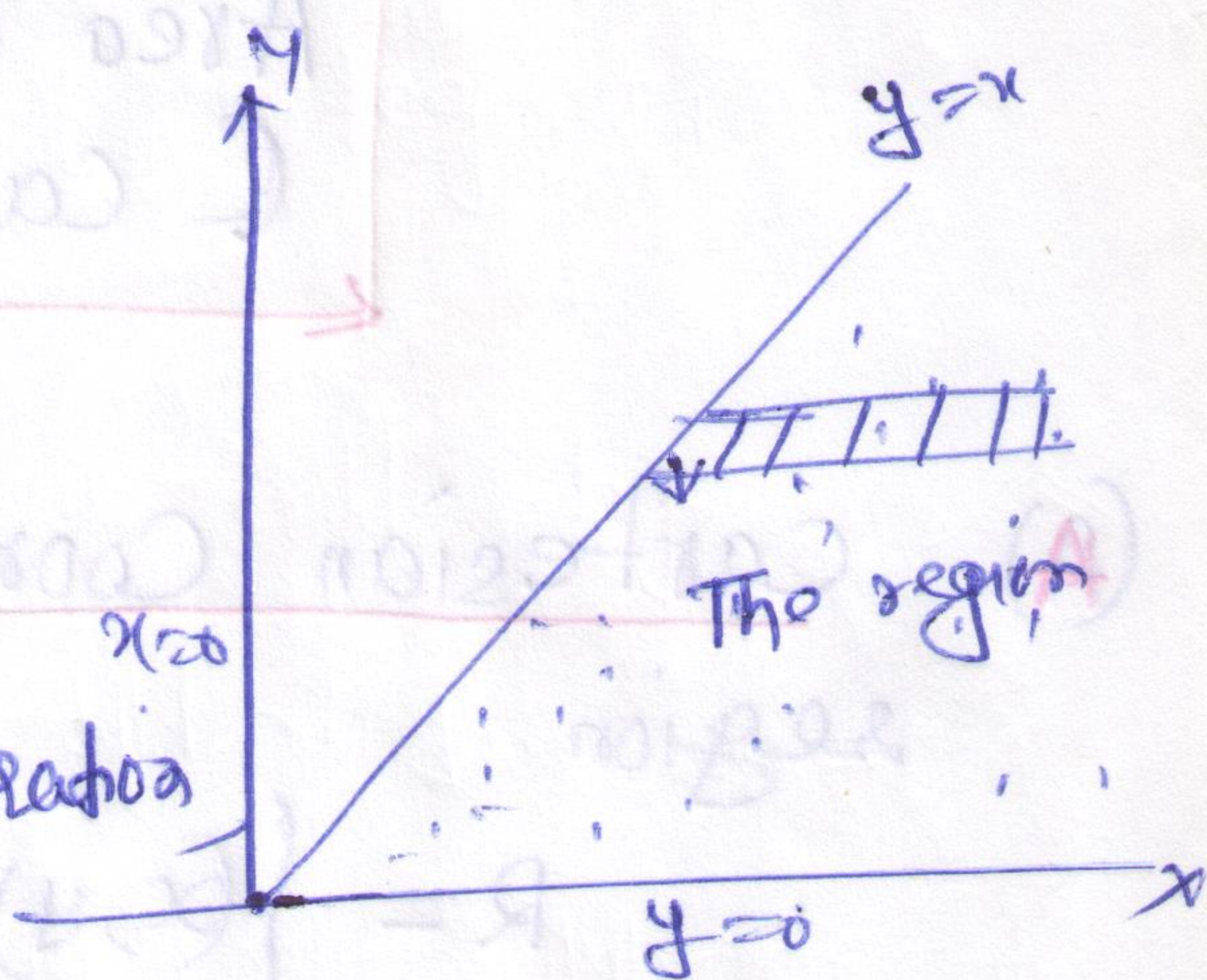
$$x=0, x=\infty, \underline{y=0}, \underline{y=x}.$$

The region of integration is as shown in the figures

On changing the order of integration

we first integrate w.r.t. (x) and then w.r.t. (y). Hence

$$\int_0^{\infty} \int_0^x x e^{-\frac{x^2}{y}} dy dx = \int_0^{\infty} \left[\int_y^{\infty} x e^{-\frac{x^2}{y}} dx \right] dy \quad (1)$$



First evaluate the inner integral.

Substituting $x^2 = t$, $2x dx = dt$,

when $x = y$, $t = y^2$ and $x = \infty$, $t = \infty$

$$\int_y^\infty x e^{-\frac{x^2}{y}} dx = \frac{1}{2} \int_{y^2}^\infty e^{-\frac{t}{y}} dt = \frac{1}{2} \left[\frac{e^{-\frac{t}{y}}}{-\frac{1}{y}} \right]_{y^2}^\infty$$

$$= \frac{1}{2} y e^{-y}.$$

Eq. (1) reduces to

$$\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dy dx = \frac{1}{2} \int_0^\infty y e^{-y} dy \quad (\text{Gamma fun})$$

$$= \frac{1}{2} \int_0^\infty y^{2-1} e^{-y} dy = \frac{1}{2} \Gamma(2) = \frac{1}{2}.$$

$$(\because \Gamma(n+1) = n!)$$

Double Integrals - change of Variables

$$\int_a^b f(x) dx = \int_c^d f(g(t)) g'(t) dt$$

Substitute: $x = g(t)$

$$a = g(c)$$

$$b = g(d)$$

~~Two variables:~~ $\iint_R f(x, y) dx dy$

Substitute $x = \phi(u, v)$
 $y = \psi(u, v)$

$$\iint_R f(x, y) dx dy = \iint_{R'} f(\phi(u, v), \psi(u, v)) |J| du dv$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

R' is the region in uv plane corresponds to R in the xy plane.

Special case:

Cartesian $\rightarrow$ Polar

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r$$

$$\therefore \iint_R f(x, y) \, dx \, dy = \iint_{R'} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$$

Example: Find the volume of a sphere in one octant. The radius of the sphere is a .

Solution:

$$V = \iint_S \sqrt{a^2 - x^2 - y^2} \, dx \, dy$$

S is the first quadrant of the circular disc $x^2 + y^2 \leq a^2$

$$\begin{aligned} \text{Put } x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$|J| = r$$

$$r \rightarrow 0 \text{ to } a$$

$$\theta \rightarrow 0 \text{ to } \frac{\pi}{2}$$

$$\therefore V = \iint \sqrt{a^2 - r^2} \, r \, dr \, d\theta$$

$$\begin{aligned} & \int_0^{\pi/2} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta \\ & \theta=0 \quad r=0 \end{aligned}$$

$$= \frac{\pi}{2} (a^2 - r^2)^{3/2} \left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) \bigg|_0^a = \frac{\pi a^3}{6}$$

Example:

$$\iint_S e^{y-x/y+x} dx dy$$

$$S: x=0, y=0, x+y=2$$

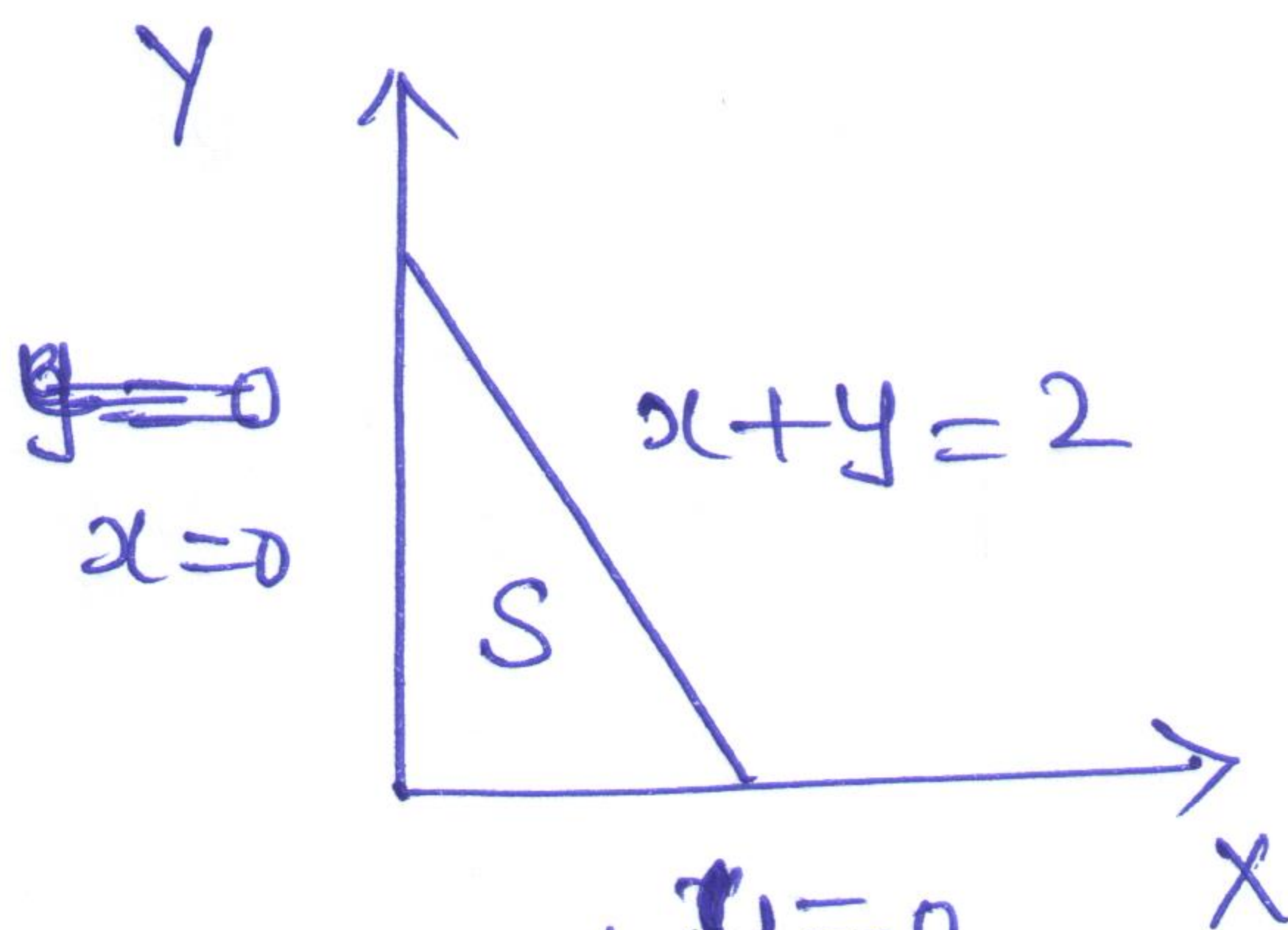
Solution:

$$\text{Put } y-x=u$$

$$y+x=v$$

$$x = \frac{v-u}{2}$$

$$y = \frac{v+u}{2}$$

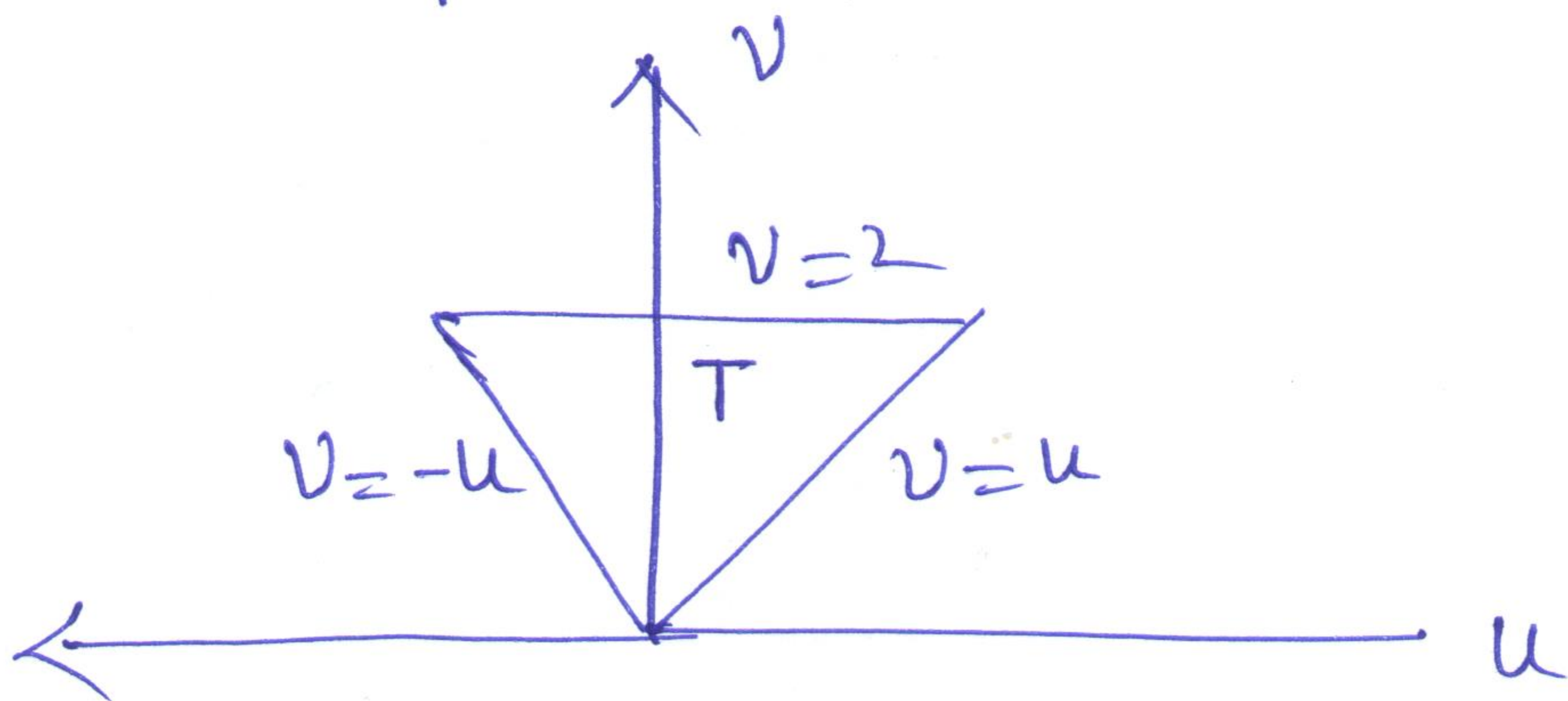


$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2}$$

$x=0$ maps to $v-u=0$ i.e. $v=u$

$y=0$ maps to $v=-u$

$x+y=2$ maps to $v=2$



$$\iint_S e^{\frac{y-x}{y+x}} dx dy = \iint_T e^{u/v} \frac{1}{2} du dv$$

$$= \frac{1}{2} \int_{v=0}^2 \int_{u=-v}^v e^{u/v} du dv$$

$$= \frac{1}{2} \int_0^2 v \left(e - \frac{1}{e} \right) dv$$

$$= e - \frac{1}{e}$$

Exc: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2) dy dx$

by changing to polar coordinates.

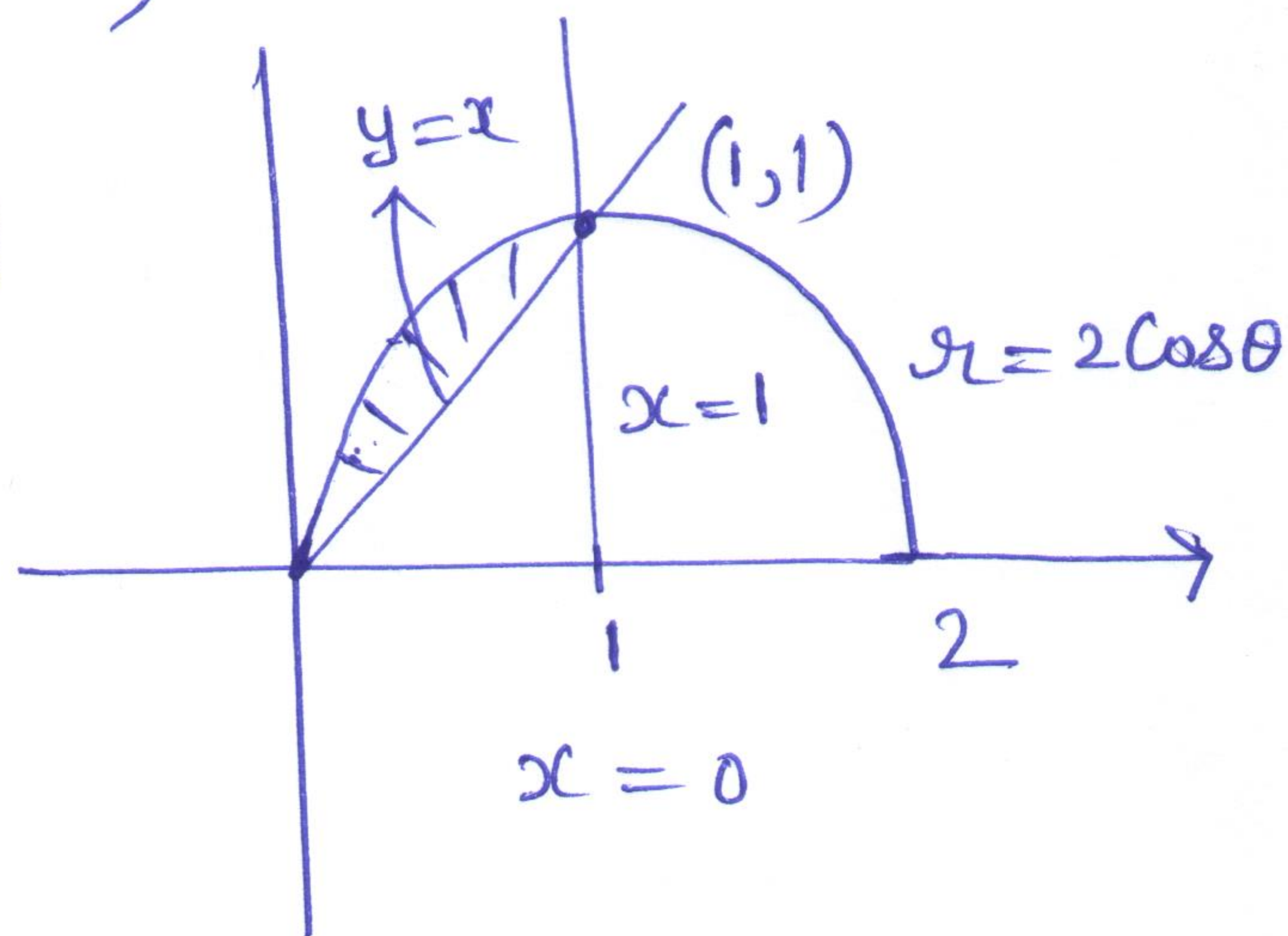
The region of integration is bounded

by $y=x$, $y=\sqrt{2x-x^2}$, $x=0$ and $x=1$.

Solution: $(x-1)^2 + y^2 = 1$

Polar equation of the circle

$$(r \cos \theta - 1)^2 + r^2 \sin^2 \theta = 1$$



$$\Rightarrow r^2 - 2r \cos \theta = 0$$

$$\Rightarrow r = 2 \cos \theta$$

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2) dy dx =$$

$$\int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2 \cos \theta} r^2 \cdot r dr d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left. \frac{r^4}{4} \right|_0^{2 \cos \theta} d\theta$$

$$= \int_{\pi/4}^{\pi/2} 4 \cos^4 \theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (2 \cos^2 \theta)^2 \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta)^2 \, d\theta$$

$$= \frac{\pi}{2} \int (1 + \cos^2 2\theta + 2 \cos 2\theta) \, d\theta$$

$$= \frac{\pi}{2} \int \left(1 + \frac{1 + \cos 4\theta}{2} + 2 \cos 2\theta \right) \, d\theta$$

$$= \frac{1}{8} (3\pi - 8)$$

Example: $\iint_R (x^2 + y^2)^{\frac{1}{2}} dx dy$

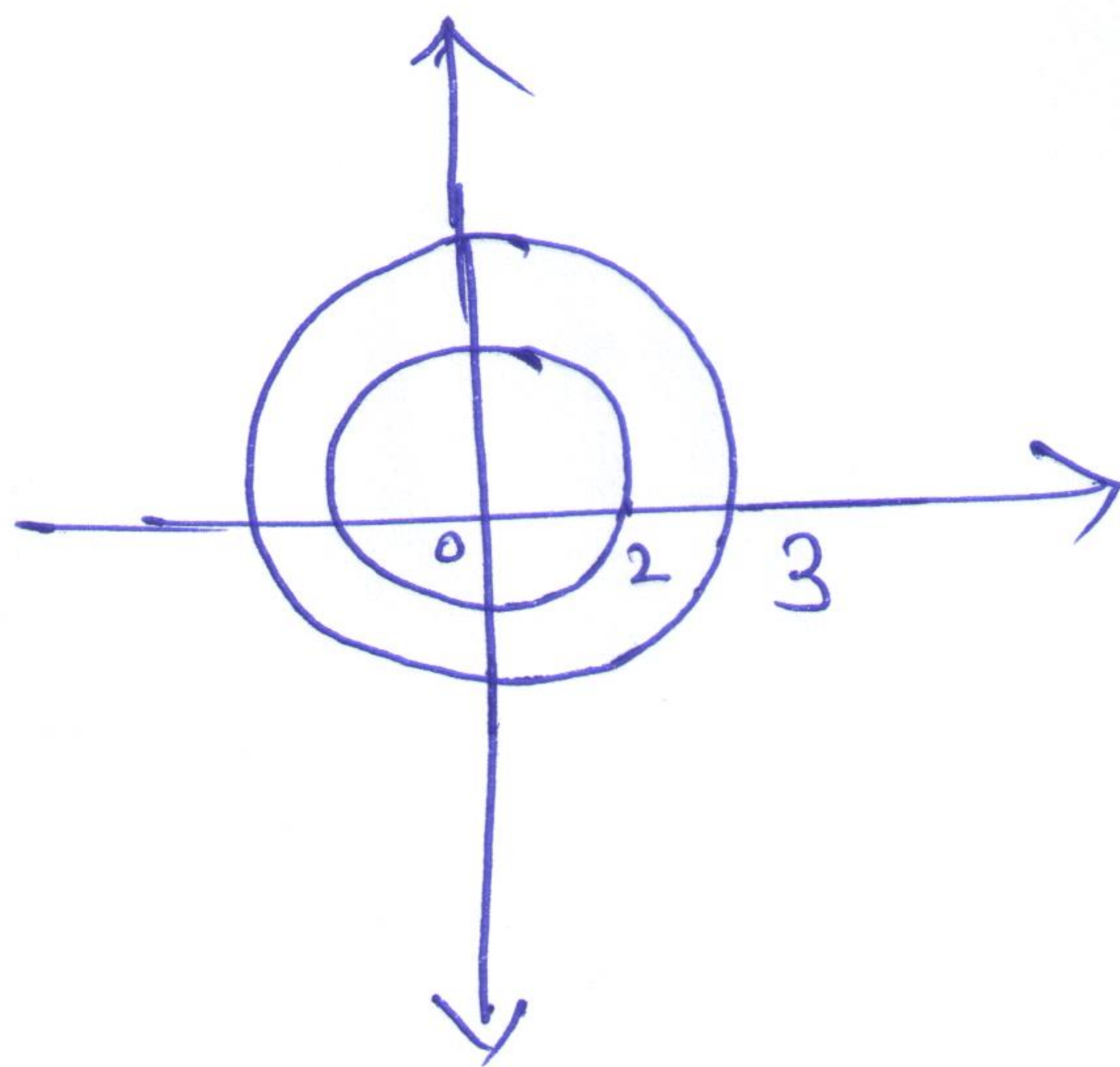
R : region in xy plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

Solution:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$|J| = r$$



$$\iint_R (x^2 + y^2)^{\frac{1}{2}} dx dy = \int_0^{2\pi} \int_2^3 (r^2)^{\frac{1}{2}} r dr d\theta$$

$$= \int_0^{2\pi} \left. \frac{r^3}{3} \right|_2^3 d\theta$$

$$= \frac{38}{3} \pi$$

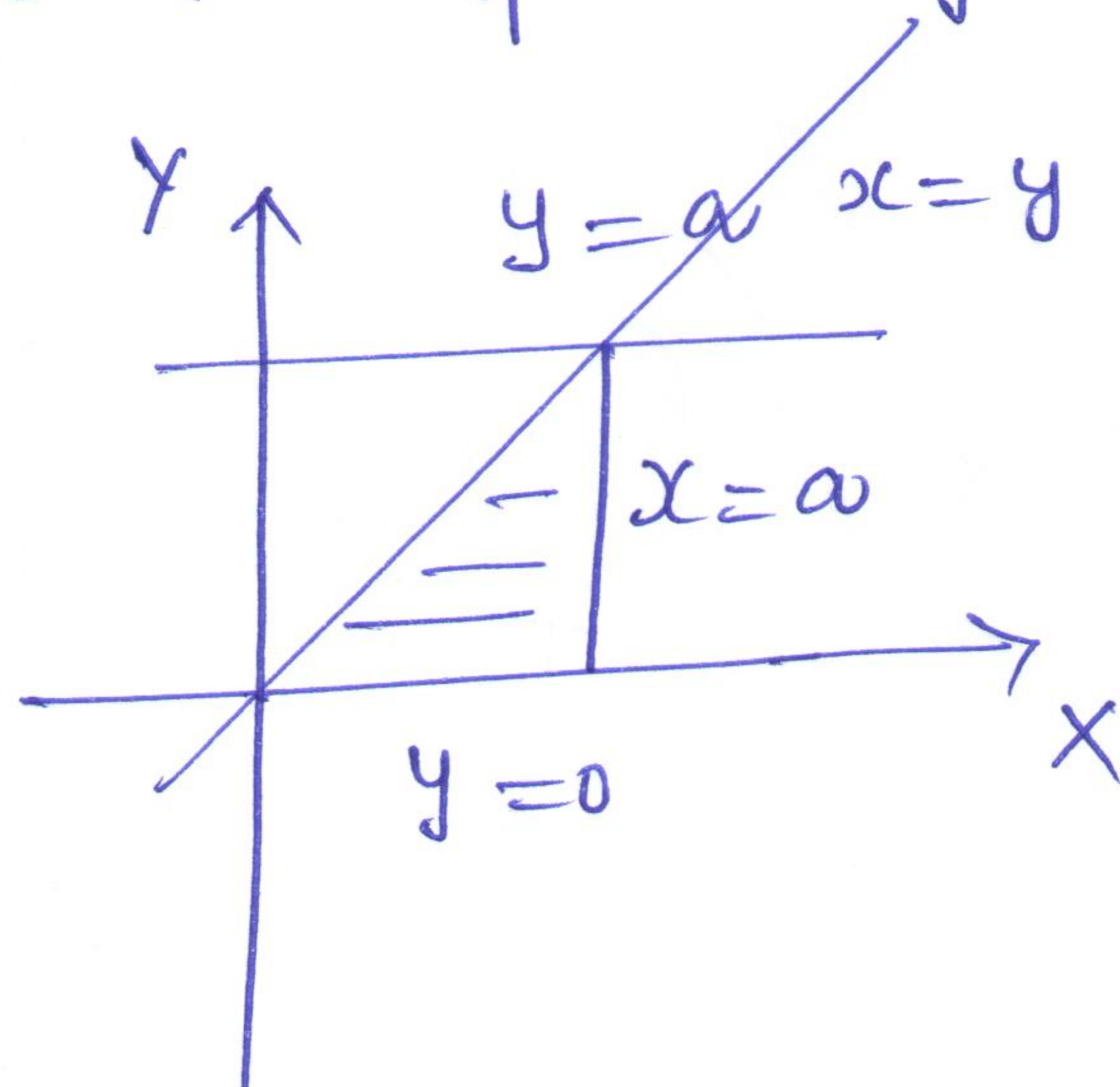
Change of order of Integration

Evaluate:

$$\int_{y=0}^a \int_{x=y}^a \frac{x}{x^2+y^2} dx dy$$

changing the order of Integration

Que: Why do we change order?



$$\int_{x=0}^a \int_{y=0}^x \frac{x}{x^2+y^2} dy dx$$

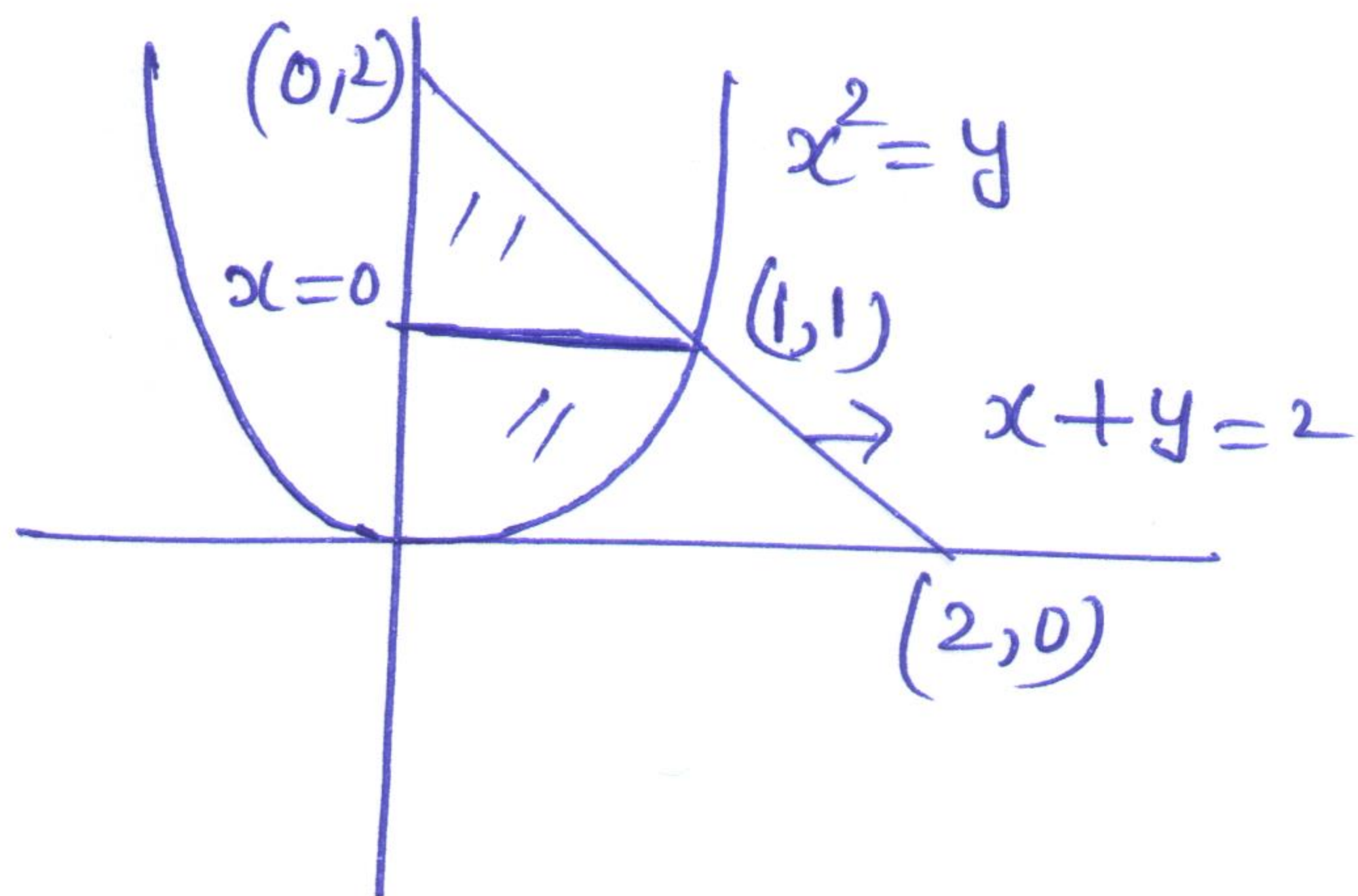
$$= \int_{x=0}^a \left[\tan^{-1} \frac{y}{x} \right]_0^x dx$$

$$= \frac{\pi}{4} \int_0^a dx = \frac{\pi a}{4}$$

Problem: Evaluate

$$\int_0^1 \int_{y=x^2}^{2-x} xy \, dy \, dx$$

by changing the order of integration.



$$\int_0^1 \int_{y=x^2}^{2-x} xy \, dy \, dx$$

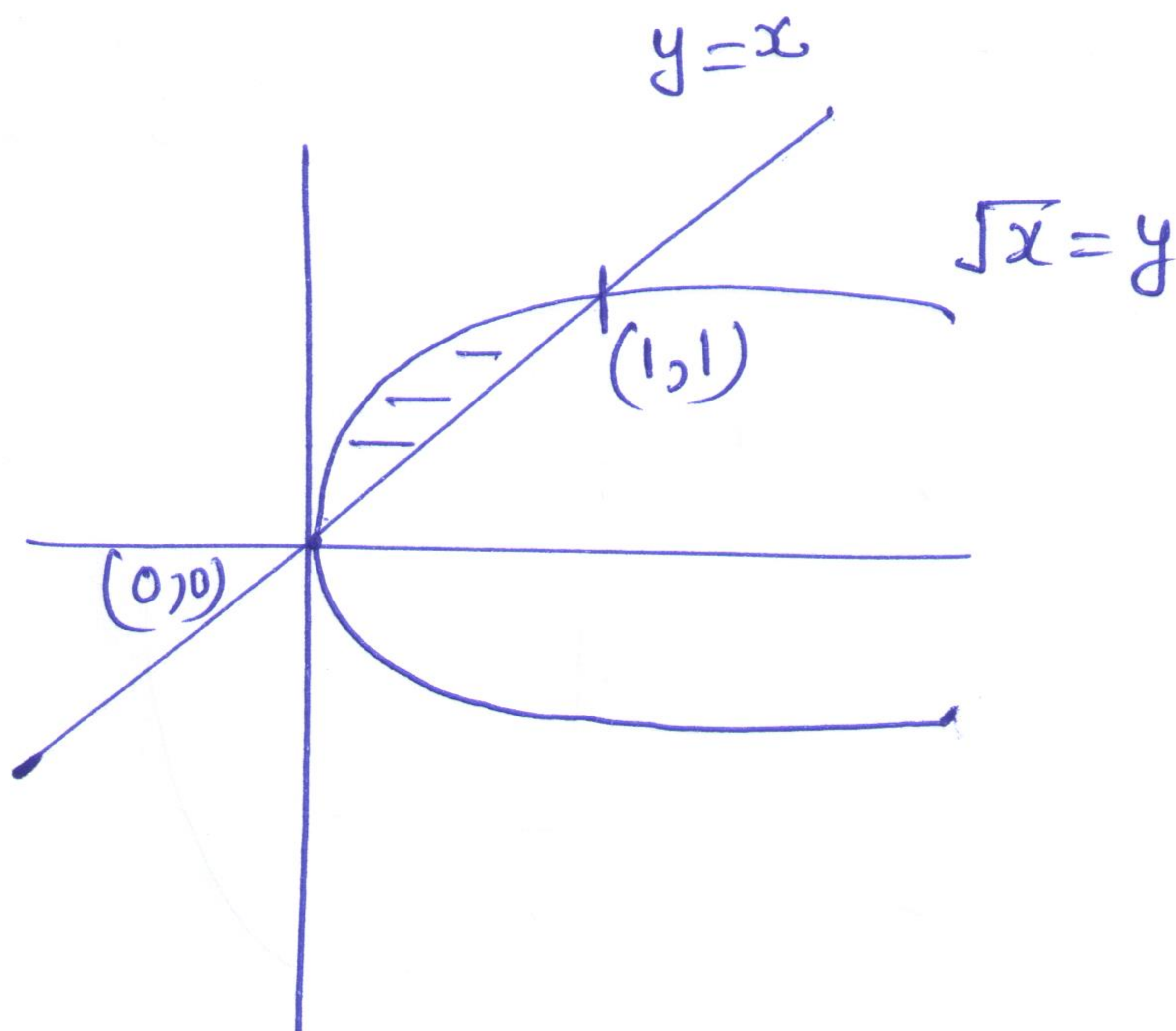
$$= \int_{y=0}^1 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^2 \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{3}{8}$$

Exc:

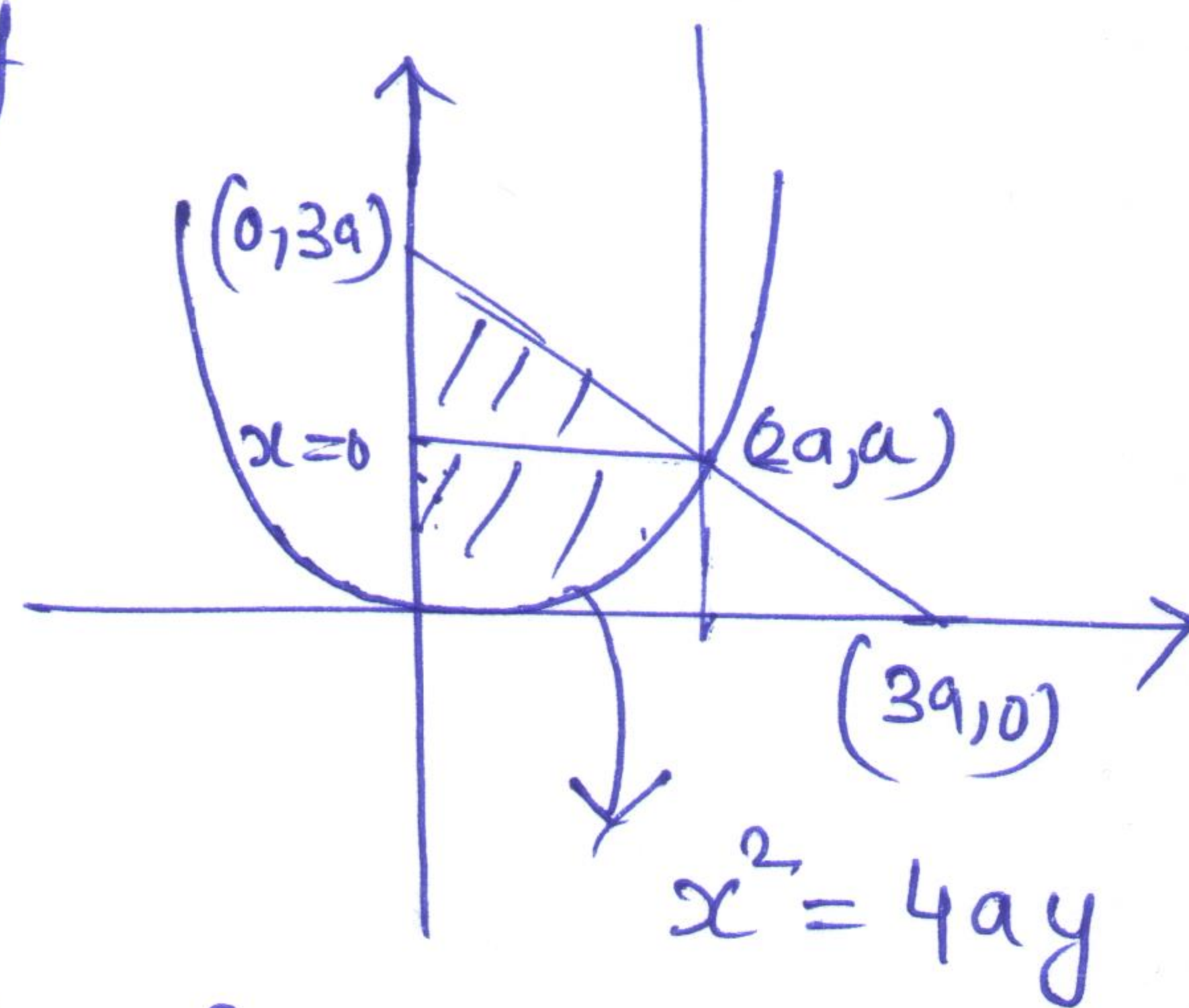
$$\int_0^1 \int_x^{\sqrt{x}} f(x, y) dy dx$$



$$= \int_0^1 \int_{y^2}^y f(x, y) dx dy$$

Exc:

$$\int_0^{2a} \int_{x^2/4a}^{3a-x} f(x, y) dy dx$$



$$= \int_0^a \int_0^{2\sqrt{ay}} f(x, y) dx dy + \int_a^{3a} \int_0^{3a-y} f(x, y) dx dy$$

Triple Integral: Divide the region V into n sub regions of respective volumes $\delta v_1, \delta v_2, \dots, \delta v_n$.

- Let (x_j, y_j, z_j) be an arbitrary point in the j th sub-region.

- Consider the sum $\sum_{j=1}^n f(x_j, y_j, z_j) \delta v_j$

If the Limit exists as $n \rightarrow \infty$ and

$\delta v_j \rightarrow 0$, then

$$\iiint_V f(x, y, z) dv = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(x_j, y_j, z_j) \delta v_j$$

- If $f(x, y, z) = 1$, then

$$\iiint_V dv = \text{Volume}$$

Evaluation of Triple Integral

$$\iiint_V f(x, y, z) \, dV = \int_{z=a}^b \left\{ \int_{y=\phi_1(z)}^{\phi_2(z)} \left[\int_{x=\phi_1(y, z)}^{\phi_2(y, z)} f(x, y, z) \, dx \right] dy \right\} dz$$

Note: Similar to deduce integrals, the order of integration is immaterial if the limits of integration are constants.

$$\begin{aligned} \int_a^b \int_c^d \int_e^f f(x, y, z) \, dx \, dy \, dz \\ = \int_e^f \int_c^d \int_a^b f(x, y, z) \, dz \, dy \, dx \\ = \int_c^d \int_e^f \int_a^b f(x, y, z) \, dz \, dy \, dx \end{aligned}$$

Problem 1. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

$$I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dy dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y} \right) dy dx$$

$$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y} \right]_0^x dx$$

$$= \frac{1}{2} \int_0^a \left(e^{4x} - 3e^{2x} + 2e^x \right) dx$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \frac{3}{8}$$

Problem 2: Evaluate $\iiint_R \frac{dx dy dz}{(x+y+z+1)^3}$

R is the region bounded by
 $x=0, y=0, z=0$ & $x+y+z=1$.

Solution: $I = \iiint_R \frac{dx dy dz}{(x+y+z+1)^3}$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz dy dx$$

$$= \int_0^1 \int_0^{1-x} \left. -\frac{1}{2} (x+y+z+1)^{-2} \right|_0^{1-x-y} dy dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (x+y+1)^{-2} - \frac{1}{4} dy dx$$

$$= \frac{1}{2} \left[\log 2 - \frac{5}{8} \right]$$

Problem: Using the triple integral find the volume common to a sphere $x^2 + y^2 + z^2 = a^2$ and a circular cylinder $x^2 + y^2 = ax$.

Solution:

$$V = \iiint_V dx dy dz$$

$$= \iiint_V dz dy dx$$

$$= 4 \int_{x=0}^a \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

$$= 4 \int_0^a \int_0^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2} dy dx$$

$$= 4 \int_0^{\pi/2} \int_{x=0}^{a \cos \theta} \sqrt{a^2-x^2} x dx d\theta$$

(233)

$$= 4 \left(-\frac{1}{2}\right) \left(\frac{2}{3}\right) \int_0^{\pi/2} (a^2 - x^2)^{3/2} \Big|_0^{a \cos \theta} d\theta$$

$$= -\frac{4a^3}{3} \int_0^{\pi/2} (\sin^3 \theta - 1) d\theta$$

$$= \frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right)$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

Change of Variables in Triple Integrals

Cartesian Coordinates to spherical
polar Coordinates

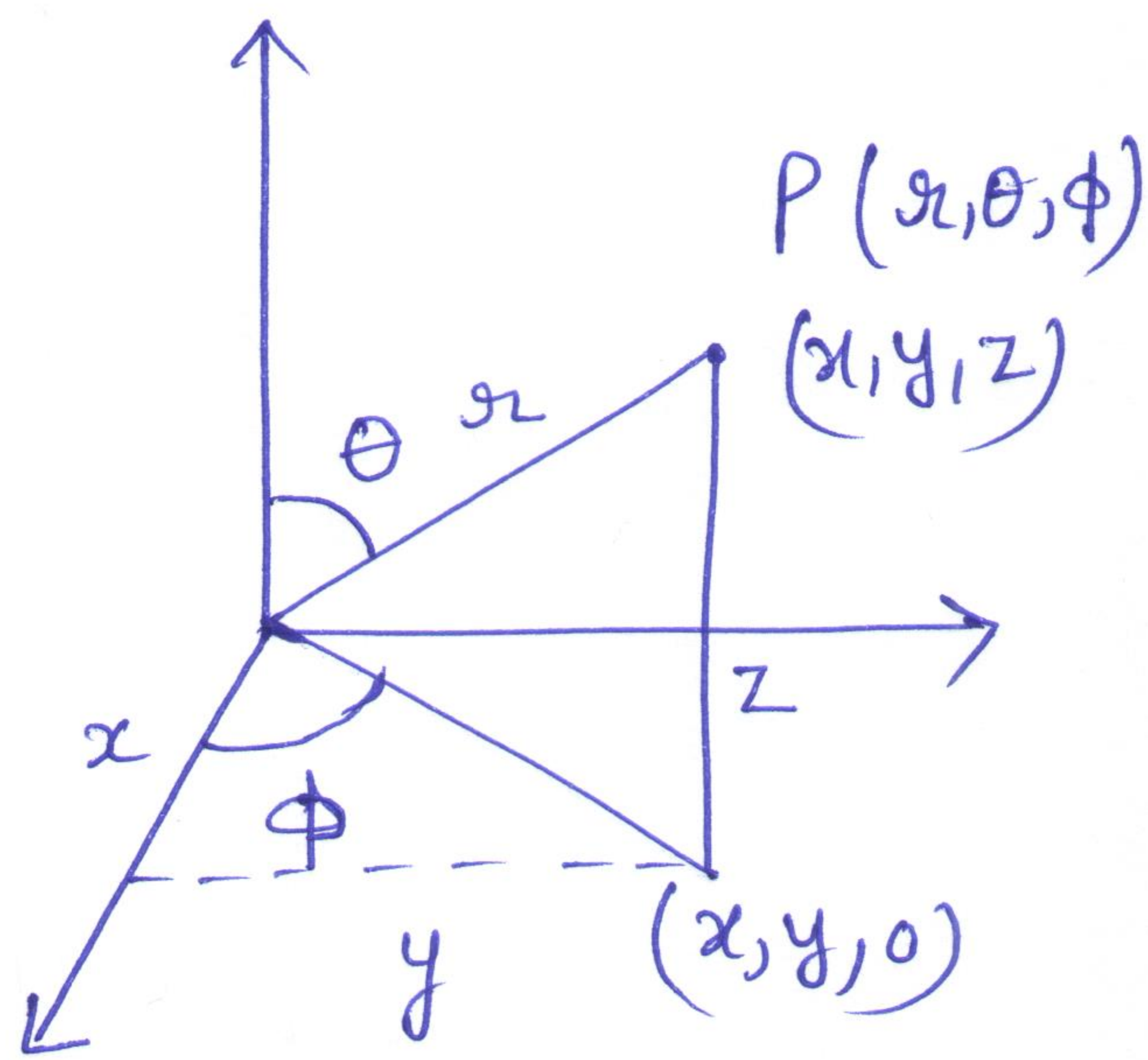
$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Note that

$$x^2 + y^2 + z^2 = r^2$$



$$\iiint_D f(x, y, z) dx dy dz =$$

$$\int \int \int_{\hat{D}} f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) |J| \cdot dr d\theta d\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta \cos\phi & r \cos\theta \cos\phi & -r \sin\theta \sin\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\theta & -r \sin\theta & 0 \end{vmatrix}$$

$$= r^2 \sin\theta$$

$$\iiint_D f(x, y, z) \, dx \, dy \, dz$$

D

$$= \iiint_{\hat{D}} f(r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Cartesian coordinates to cylindrical coordinates

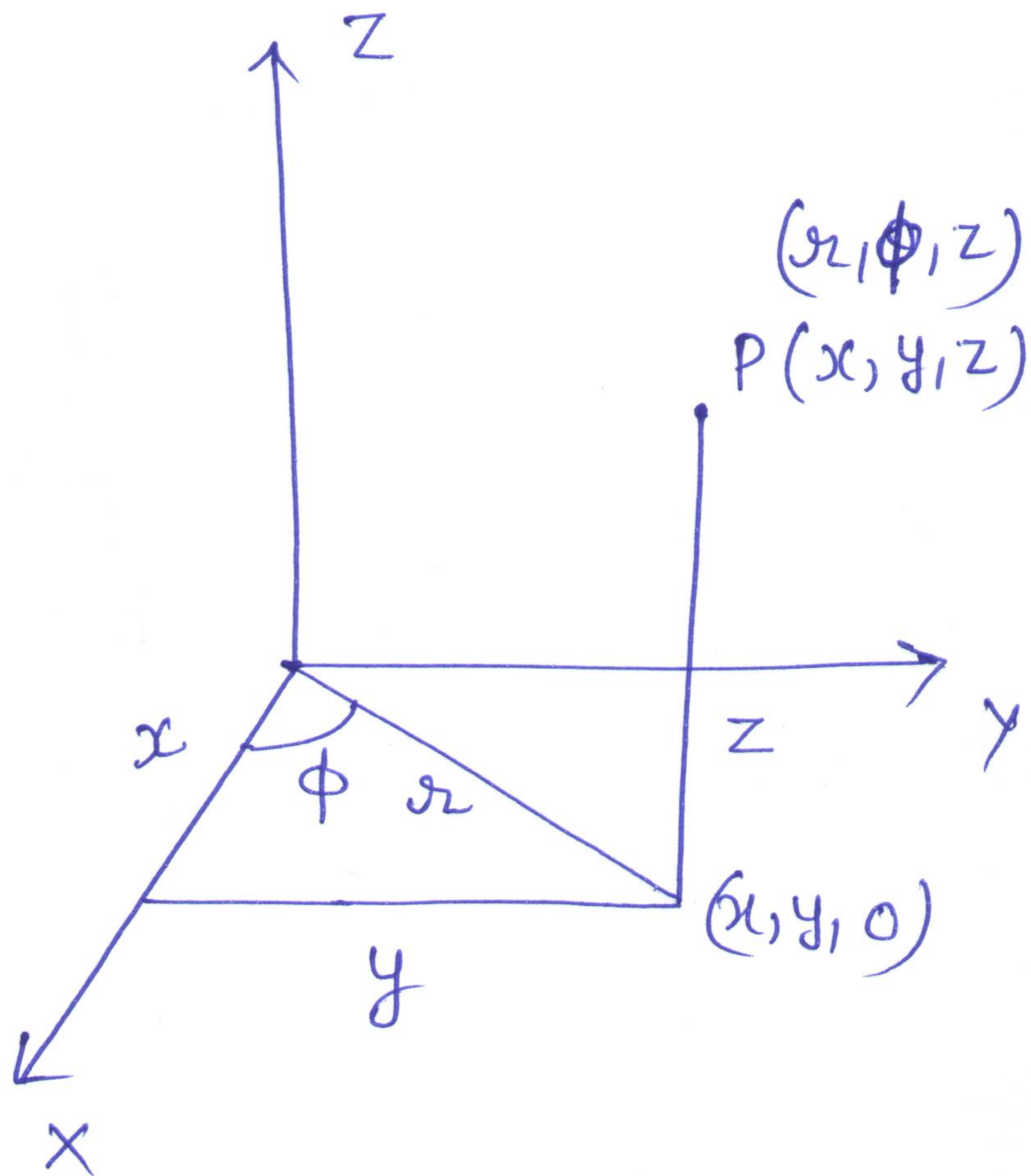
$$x = r \cos \phi$$

$$y = r \sin \phi$$

$$z = z$$

Note that

$$x^2 + y^2 = r^2$$



$$\iiint f(x, y, z) \, dx \, dy \, dz$$

$$\stackrel{D}{=} \iiint_{\hat{D}} f(r \cos \phi, r \sin \phi, z) |J| \, dr \, d\phi \, dz$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \phi & -r \sin \phi & 0 \\ \sin \phi & r \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Problem 1. changing to cylindrical
Coordinate, evaluate

$$\iiint_D z(x^2 + y^2) dx dy dz$$

D

$$D: x^2 + y^2 \leq 1, 2 \leq z \leq 3$$

Solution:

$$x = r \cos \theta$$

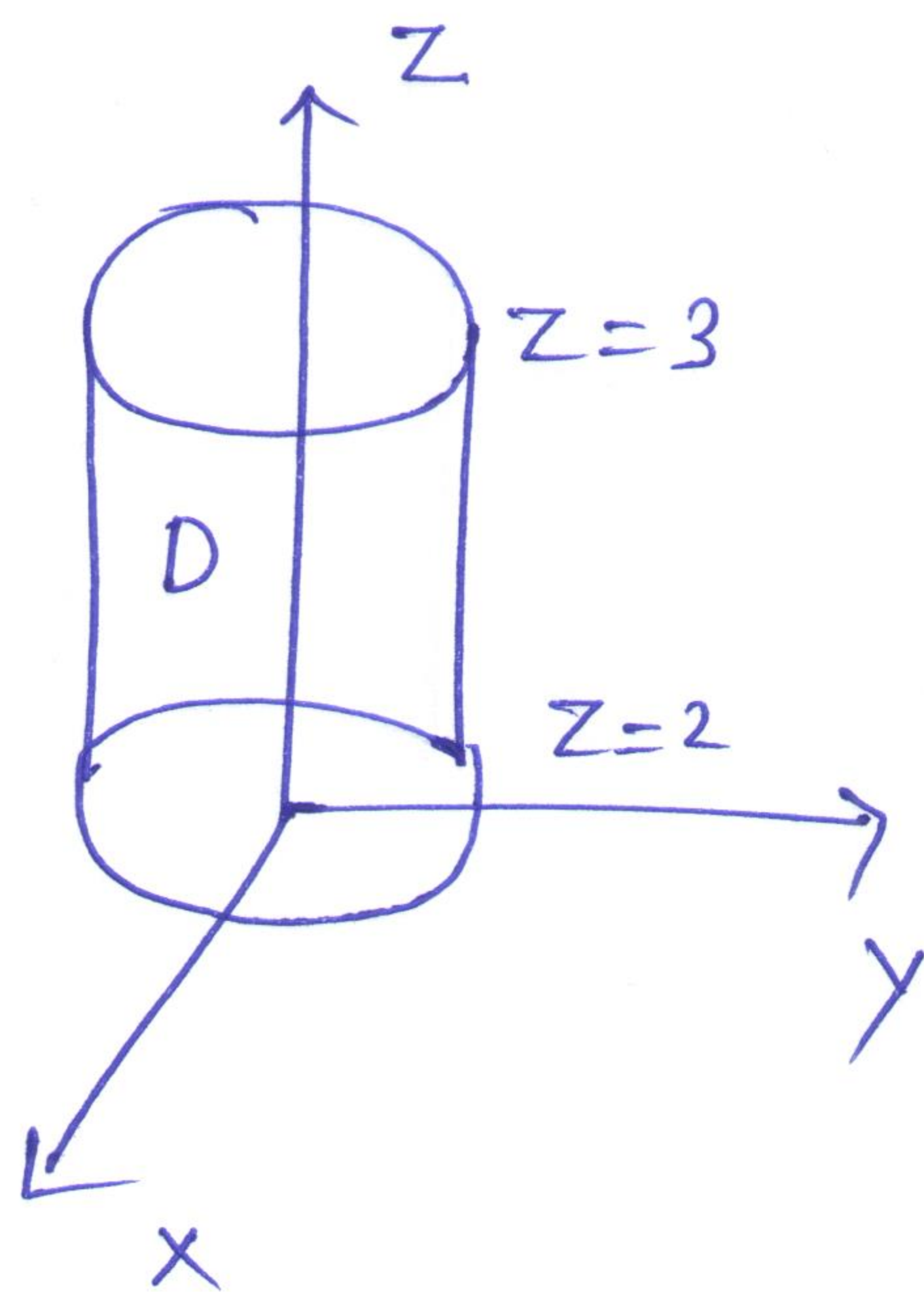
$$y = r \sin \theta$$

$$z = z$$

$$x^2 + y^2 = r^2, \quad J = r$$

$$I = \int_{z=2}^3 \int_{\theta=0}^{2\pi} \int_{r=0}^1 z \cdot r^2 \cdot r dr d\theta dz$$

$$= \frac{5\pi}{4}$$



Exc: Change to spherical coordinate

evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}} dz dy dx$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

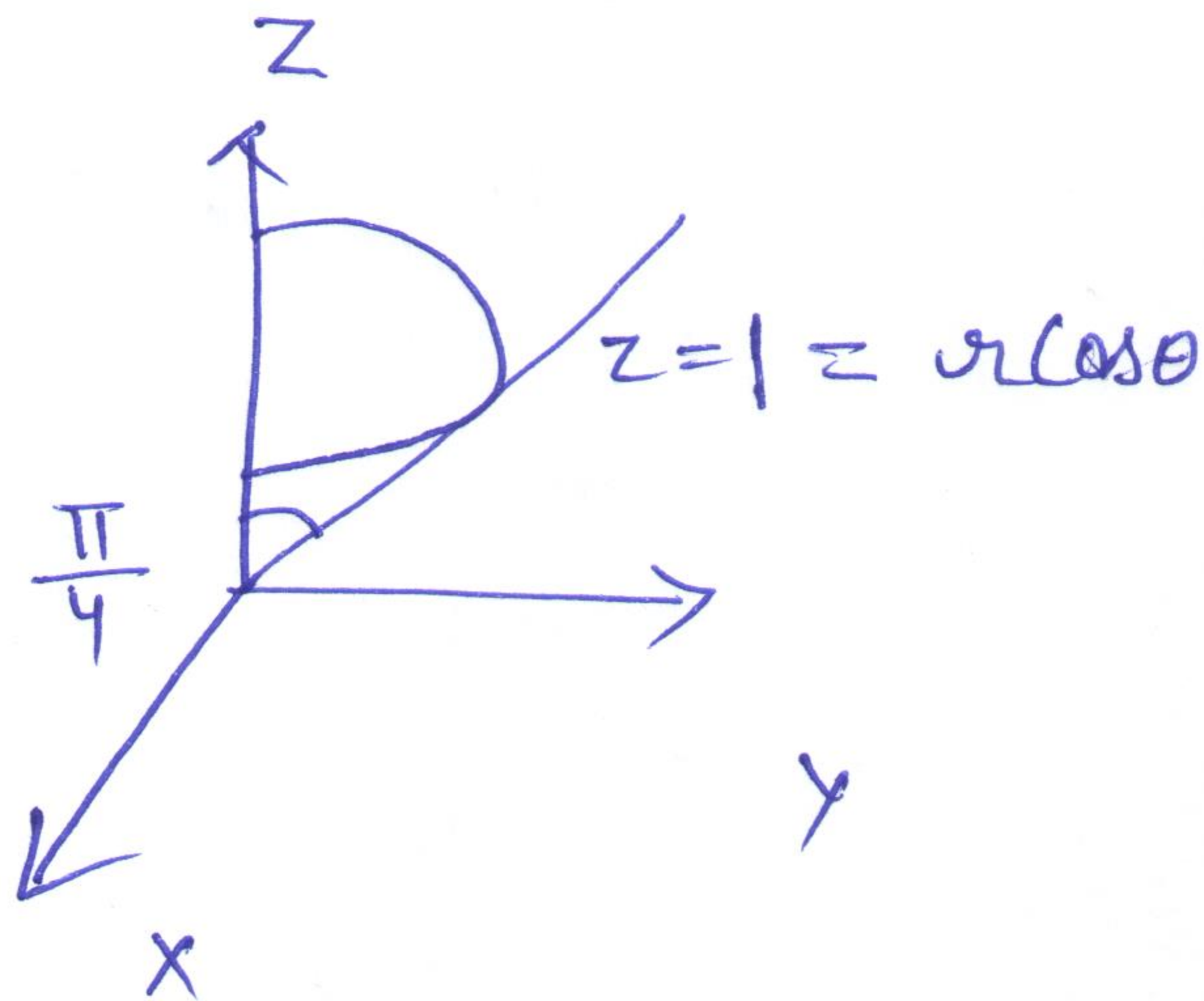
$$z = r \cos \theta$$

$$J = r^2 \sin \theta$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}} dz dy dx$$

$$= \int_0^{\pi/2} \int_0^{\pi/4} \int_{r=0}^{\sec \theta} \frac{1}{r} r^2 \sin \theta dr d\theta d\phi$$

$$= \frac{(\sqrt{2}-1)\pi}{4}$$



Exc: Change to spherical coordinates, evaluate

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dz dy dx$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$J = r^2 \sin \theta, \quad x^2 + y^2 + z^2 = r^2$$

$$\therefore I = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \int_{r=0}^1 \frac{r^2 \sin \theta}{\sqrt{1-r^2}} dr d\theta d\phi$$

$$= \frac{\pi^2}{8}$$

Multiple Integrals

(1)

The double integral of $f(x, y)$ over the region

R is denoted by

$$\iint_R f(x, y) dR.$$

$$\lim_{\substack{n \rightarrow \infty \\ \delta A_r \rightarrow 0}} \sum_{k=1}^n f(x_k, y_k) \delta A_r = \iint_R f(x, y) dA$$

$\delta A_r \rightarrow \text{Area of } r\text{th rectangle}$

Theorem: (i) Let ϕ and ψ be two continuous functions defined on $[a, b]$ such that $\phi(x) \leq \psi(x)$ for all $x \in [a, b]$. Let f be a cts. func. defined over $R = \{(x, y) : a \leq x \leq b, \phi(x) \leq y \leq \psi(x)\}$.

Then

$$\iint_R f(x, y) dx dy \quad \text{and} \quad \int_a^b \left[\int_{\phi(x)}^{\psi(x)} f(x, y) dy \right] dx$$

exist and are equal.

(ii) Let ϕ and ψ be cts. func. on $[c, d]$ s.t. $\phi(y) \leq \psi(y)$ for all $y \in [c, d]$. Let f be a cts. func. defined over $R = \{(x, y) : c \leq y \leq d, \phi(y) \leq x \leq \psi(y)\}$.

Then

$$\iint_R f(x, y) dx dy \quad \text{and} \quad \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x, y) dx \right] dy$$

exist and are equal.

Ex. Show that

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dx \right] dy \neq \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dy \right] dx.$$

L.H.S.

$$\begin{aligned} \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx \right) dy &= \int_0^1 \left(\int_0^1 \frac{x+y-2y}{(x+y)^3} dx \right) dy \\ &= \int_0^1 \left[\int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right) dx \right] dy = \int_0^1 \left[\frac{1}{1+y} + \frac{2y}{2(1+y)^2} \right] dy \\ &= \int_0^1 \frac{1}{(1+y)^2} dy = \left[\frac{1}{1+y} \right]_0^1 = -\frac{1}{2} \end{aligned}$$

Similarly, we can show R.H.S

$$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} dy \right] dx = \frac{1}{2}.$$

Hence, L.H.S $\neq$ R.H.S.

The reason is that the function $f = \frac{x-y}{(x+y)^3}$ is not cts in $R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Note: Both exist but not equal.

(3)

Evaluation of Double Integral (Cartesian Coordinates)

- (i) If the limits in the inner integral are funcs. of x , then we evaluate $\int_{\phi(x)}^{\psi(x)} f(x,y) dy$ first, taking x as a constant and evaluate the integrand (function of x) obtained in the first step, integrating it w.r.t. x between a and b . Thus we have

$$\int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y) dx dy = \int_a^b \left[\int_{y=\phi(x)}^{y=\psi(x)} f(x,y) dy \right] dx.$$

- (ii) Similarly,

$$\int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y) dx dy = \int_c^d \left[\int_{\phi(y)}^{\psi(y)} f(x,y) dx \right] dy.$$

Ex. 1: $\int_1^{\log 8} \int_{x=0}^{\log y} e^{x+y} dx dy$

$$\begin{aligned} I &= \int_1^{\log 8} \left(\int_0^{\log y} e^x dx \right) e^y dy = \int_1^{\log 8} [e^x]_0^{\log y} e^y dy \\ &= \int_1^{\log 8} (e^{\log y} - 1) e^y dy = \int_1^{\log 8} e^y (y-1) dy \\ &= [e^y y - 2e^y]_1^{\log 8} = \boxed{8 \log 8 - 16 + e} \end{aligned}$$

Ex 2: evaluate $\int_0^\pi \int_0^x \sin y \, dy \, dx$.

4

Hint: $\int_0^\pi \left(\int_0^x \sin y \, dy \right) dx = \pi$.

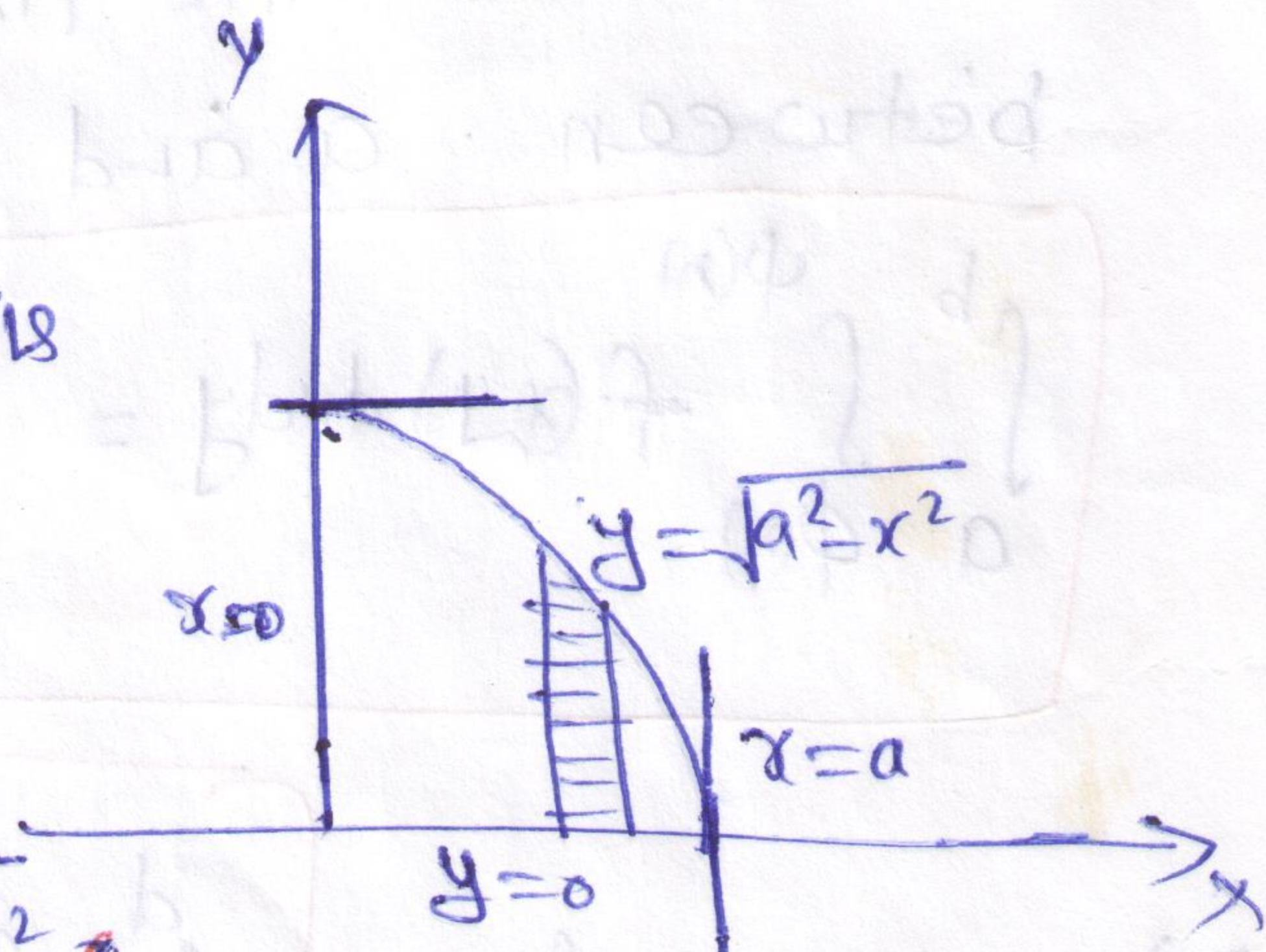
Ex 3: Evaluate $\iint_A xy \, dx \, dy$ over the +ve quadrant of the circle $x^2 + y^2 = a^2$.

Sol: Here the region of integration is

the quadrant of $x^2 + y^2 = a^2$

where x varies from "0 to a "

and y varies from 0 to $\sqrt{a^2 - x^2}$.



Hence,
$$\iint_A xy \, dx \, dy = \int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \, dx$$

$$= \int_0^a \left[\int_{y=0}^{\sqrt{a^2-x^2}} y \, dy \right] x \, dx = \int_0^a \left[\frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} x \, dx$$

$$= \frac{1}{2} \int_0^a x (a^2 - x^2) \, dx = \frac{1}{2} \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{1}{8} a^4.$$

Ex 4: Evaluate $\int_0^1 \int_0^{x^2} (x^2 + y^2) \, dx \, dy = \frac{26}{105}$.

Hint

$$R = \{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2 \}.$$

Ex. 5:

(5)

Evaluate $\iint_R xy \, dx \, dy$, R is the domain bounded by the x -axis, ordinate $x=2a$, and the curve $x^2=4ay$.

Sol: The region of integration is

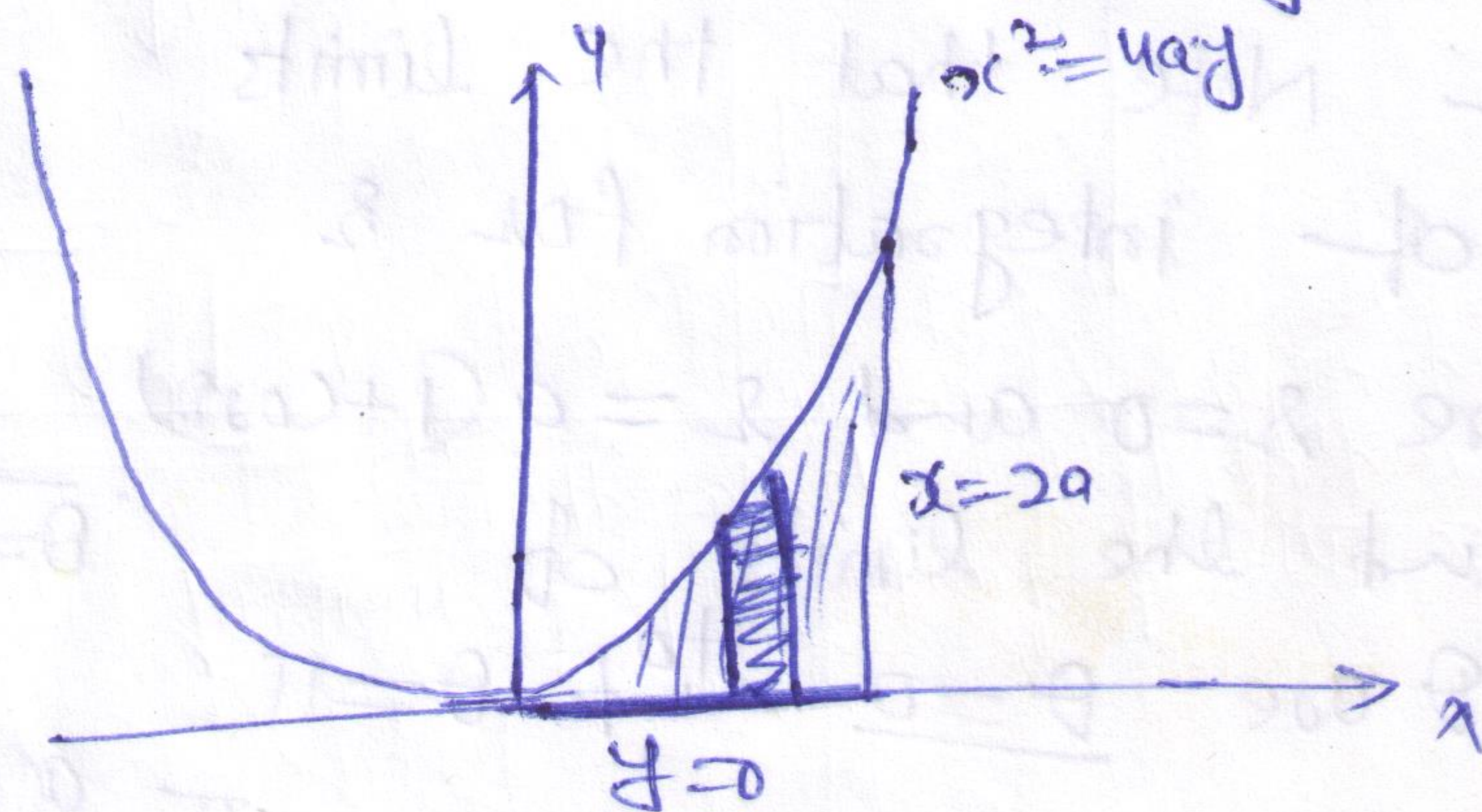
$R = \{ (x,y) : 0 \leq x \leq 2a, 0 \leq y \leq \frac{x^2}{4a} \}$. The region

is bdd. by $y=0$, $x=2a$ and parabola $x^2=4ay$

$$\iint_R xy \, dx \, dy$$

$$= \int_0^{2a} \int_0^{\frac{x^2}{4a}} xy \, dy \, dx$$

$$= \int_0^{2a} \left[\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x^5}{32a^2} dx = \frac{a^4}{3}$$



Evaluation of Double Integrals (Polar Coordinates)

We wish to evaluate

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta) \, dr \, d\theta,$$

over the region bounded by

the straight lines $\theta = \theta_1$ and $\theta = \theta_2$ and the

curves $r = r_1$ and $r = r_2$.

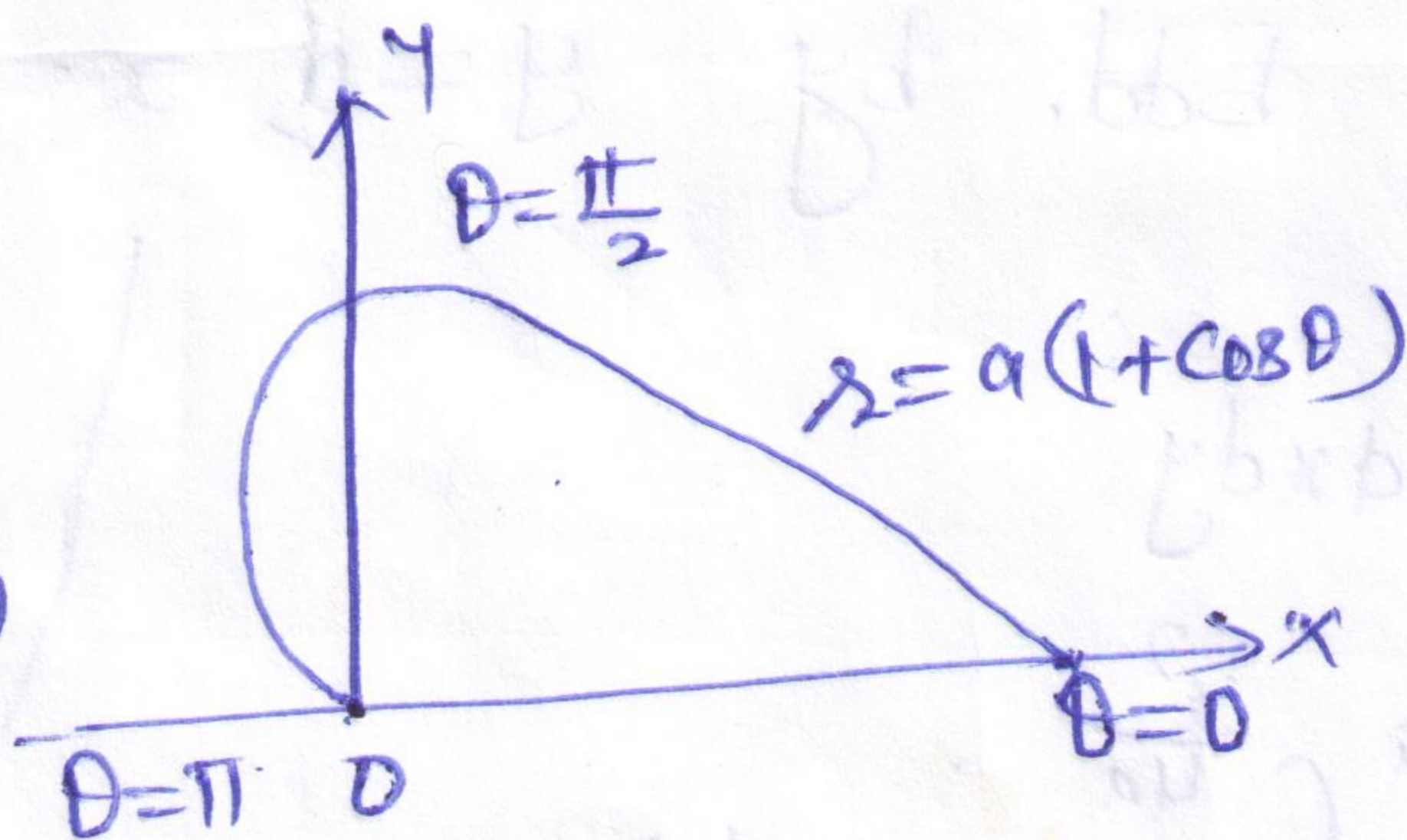
(5)

We first integrate $f(r, \theta)$ w.r.t. r between $r=r_1$ and $r=r_2$.

Then, we integrate resulting integrand w.r.t. θ between $\theta=\theta_1$ and $\theta=\theta_2$.

EX. 1: Evaluate $\iint_R r \sin \theta \, dr \, d\theta$ over the area of the cardioid $r = a(1 + \cos \theta)$ above the initial line.

SOL: Note that the limits of integration for r are $r=0$ and $r=a(1+\cos \theta)$ and the limits of θ are $\theta=0$ to $\theta=\pi$



$$\iint_R r \sin \theta \, dr \, d\theta = \int_0^\pi \int_0^{a(1+\cos \theta)} r \sin \theta \, dr \, d\theta$$

$$= \int_0^\pi \sin \theta \left[\frac{r^2}{2} \right]_0^{a(1+\cos \theta)} d\theta$$

$$= \frac{1}{2} \int_0^\pi a^2 (1+\cos \theta)^2 \sin \theta \, d\theta = \frac{a^2}{2} \int_0^\pi \left(2\cos^2 \frac{\theta}{2} \right)^2 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

$$= 4a^2 \int_0^\pi \cos^5 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta$$

$$= 4a^2 \int_0^{\pi/2} \cos^5 \phi \sin \phi \cdot 2 d\phi, \quad \theta = 2\phi$$

$$= 8a^2 \int_0^{\pi/2} \cos^5 \phi \sin \phi d\phi = -8a^2 \int_0^{\pi/2} \cos^4 \phi (-\sin \phi) d\phi$$

$$= -8a^2 \left[\frac{\cos^5 \phi}{5} \right]_0^{\pi/2} = \boxed{\frac{4a^2}{3}}$$

Change of Variables in a Double Integral

Let D be a domain in $\mathbb{R}^2$ and $f: D \rightarrow \mathbb{R}$ be integral over D . Suppose that D be mapped on to a set A of (uv -plane) by the transformation $x = \phi(u, v)$ and $y = \psi(u, v)$, ϕ, ψ have cts. partial derivatives in A and the Jacobian $\frac{\partial(x, y)}{\partial(u, v)} \neq 0 \quad \forall (u, v) \in A$. Then

$$\iint_D f(x, y) dx dy = \iint_A f(\phi, \psi) |J| du dv.$$

Deduction: If there is a change in variable from Cartesian — to polar coordinates

then $x = r \cos \theta$, $y = r \sin \theta$.

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

Thus,

$$\iint_D f(x, y) dx dy = \iint_A f(r \cos \theta, r \sin \theta) r d\theta dr.$$

steps: (1) substitute $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$
(2) supply polar limits of integration for boundary D .

(8)

Ex. 1: Calculate the integral

$$\iint \frac{(x-y)^2}{x^2+y^2} dx dy, \quad \text{over the circle } x^2+y^2 \leq 1.$$

Sol: The value of the integral is "4 times" the value of the integral taken over the +ve quadrant of the circle $x^2+y^2=1$.

Substituting $x = r \cos \theta$, $y = r \sin \theta$,
the given integral becomes:

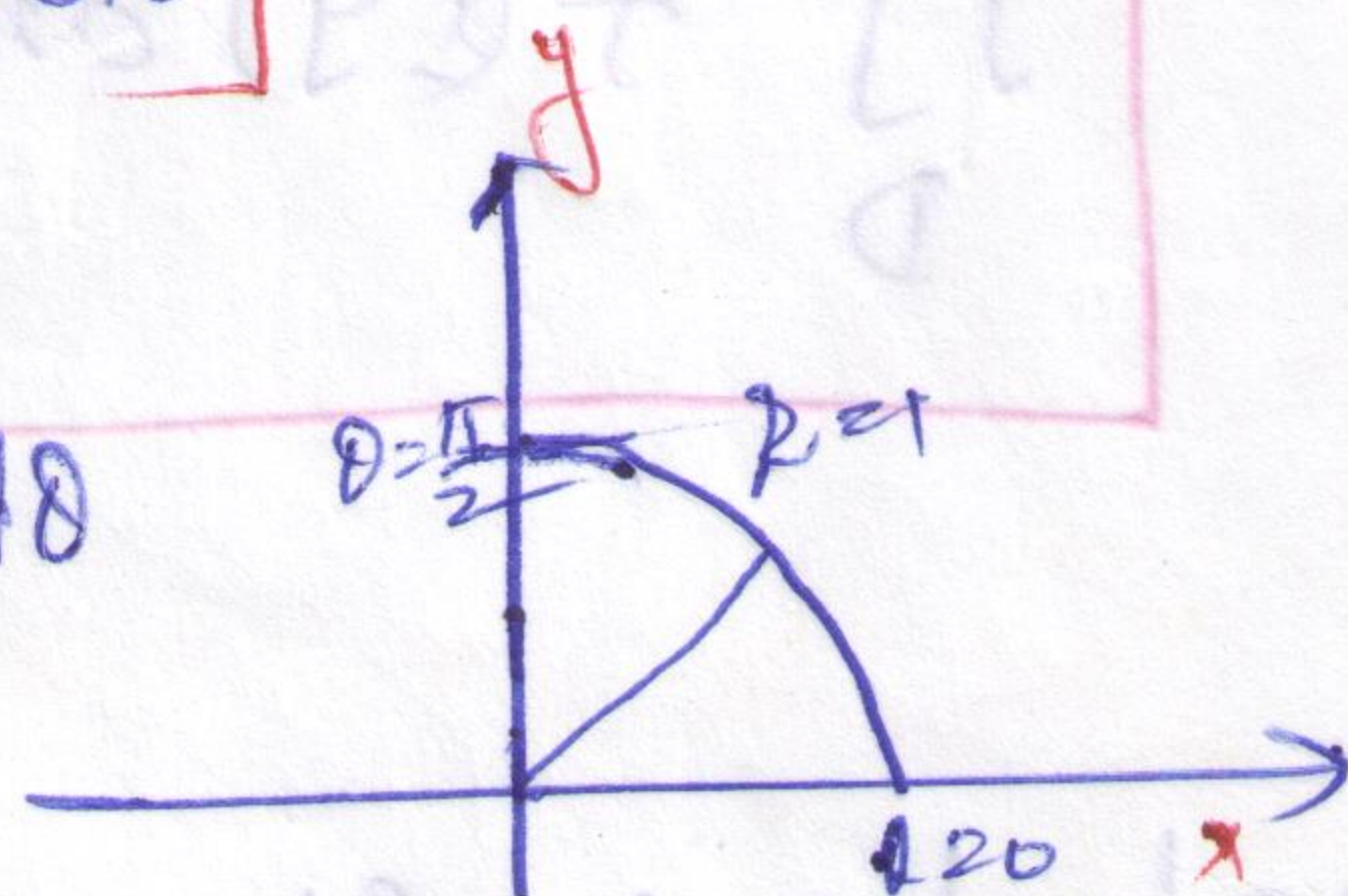
$$0 \leq r \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2}$$

$$= 4 \times \left[\int_0^1 \int_0^{\pi/2} \frac{1}{r^2} (r \cos \theta - r \sin \theta)^2 |J| dr d\theta \right]$$

$$= 4 \int_0^1 r \int_0^{\pi/2} (\cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta) dr d\theta$$

$$= 4 \int_0^{\pi/2} (1 - \sin 2\theta) \left(\int_0^1 r dr \right) d\theta$$

$$= 2 \int_0^{\pi/2} (1 - \sin 2\theta) d\theta = 2 \left[\theta + \frac{\cos 2\theta}{2} \right]_0^{\pi/2} = \boxed{\pi - 2}$$



Ex. 2 Evaluate $\iint_{x^2+y^2 \leq 1} \sin \pi (x^2+y^2) dx dy$.

Sol: Changing to polar form

$$\iint_{x^2+y^2 \leq 1} \sin \pi (x^2+y^2) dx dy = \int_0^{2\pi} \int_0^1 \sin(\pi r^2) r dr d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_0^1 \sin(\pi t) dt \right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[-\cos(\pi t) \right]_0^1 d\theta$$

$$x^2 = t$$

$$2x dx = dt$$

$$x dx = \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{2\pi} \left[-\cos \pi + \cos 0 \right] d\theta = \frac{1}{2} \int_0^{2\pi} (1+1) d\theta$$

$$= \frac{1}{2} (2\pi) = \pi.$$

Change of order of integration

If the limits of integration are variable, a change in the order of integration requires a change in the limits of integration.

Ex (1) Change the order of integration in

$$I = \int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x,y) dx dy \quad \text{--- (1)}$$

Sol: The region of integration is add by

$$y = -a, y = a, x = 0, x = \sqrt{a^2-y^2} \text{ or } x^2+y^2 = a^2.$$

We have

$$I = \int_{-a}^a \left[\int_0^{\sqrt{a^2-y^2}} f(x,y) dx \right] dy.$$

Here, we first integrate w.r.t. 'x' and then w.r.t. 'y'.

On changing the order of integration,

we first integrate w.r.t 'y', along a vertical strip RS,

which extends from $y = -\sqrt{a^2 - x^2}$

to $y = \sqrt{a^2 - x^2}$.

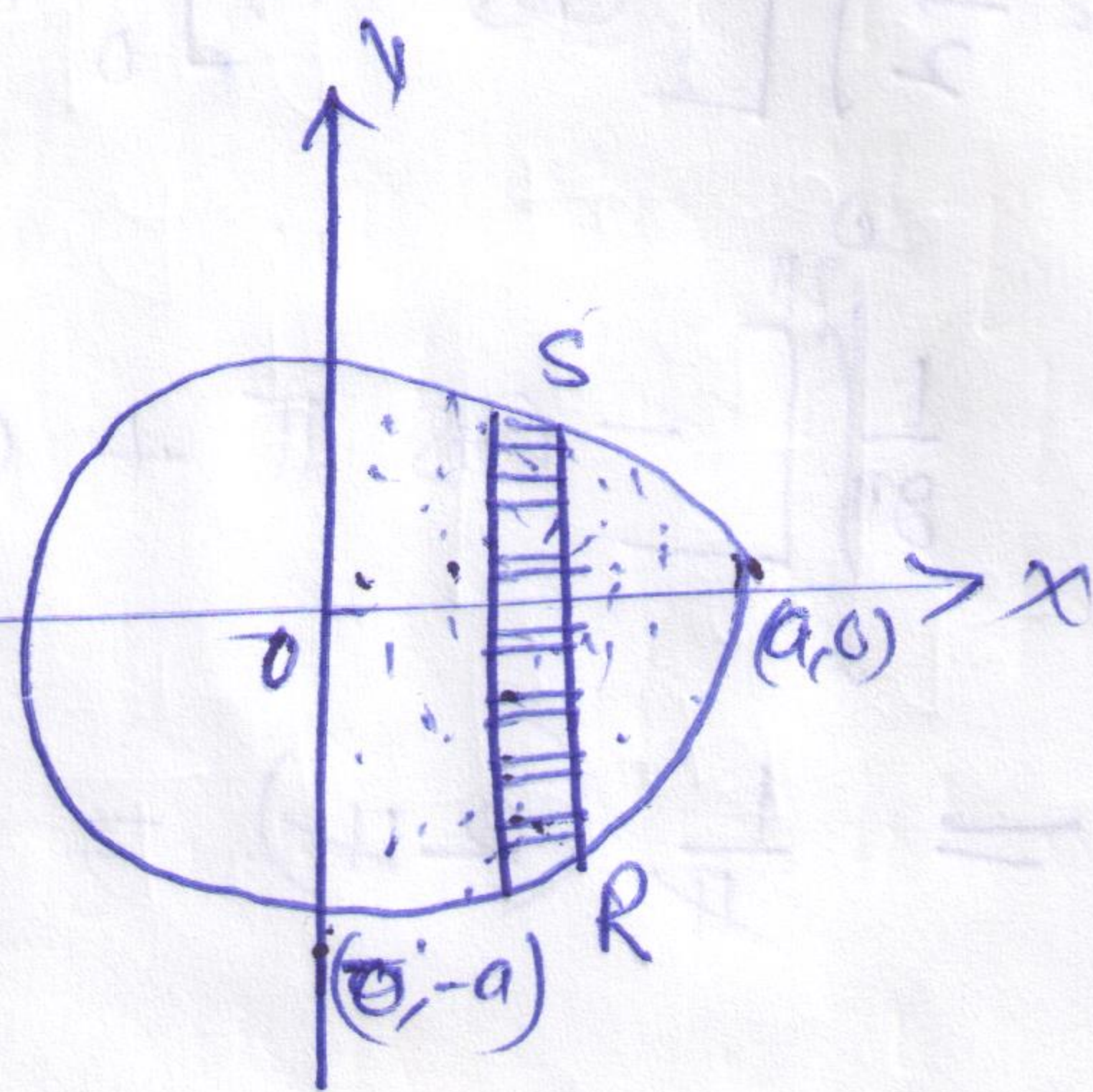
To cover the whole region

of integration, we then

integrate w.r.t x from

$x=0$ to $x=a$. Hence,

$$I = \int_0^a \left[\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y) dy \right] dx.$$



Ex. 2 Changing the order of integration, find the value of the integral

$$\int_0^{\infty} \int_x^{\infty} \frac{e^{-y}}{y^2} dy dx.$$

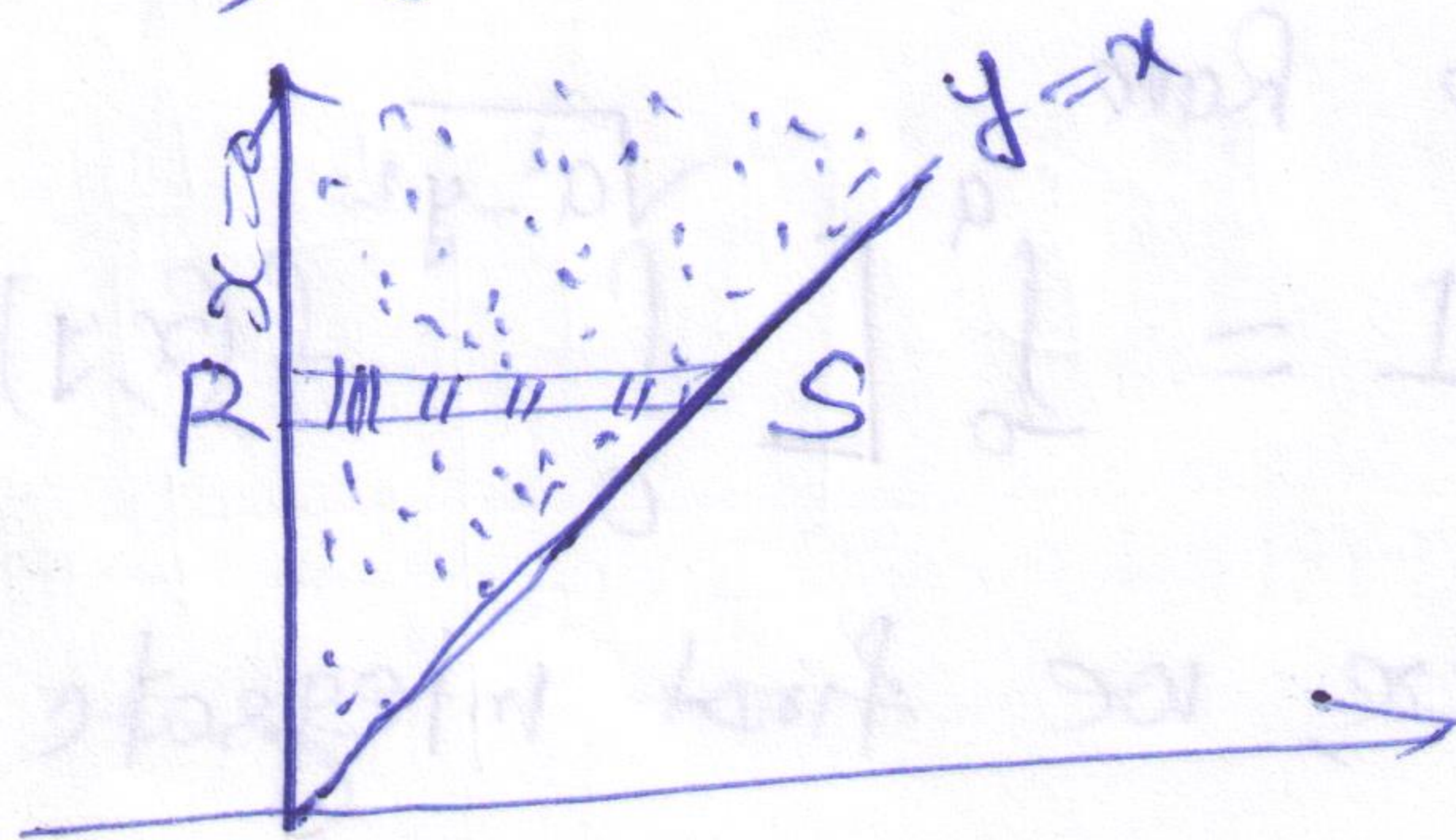
Sol: The region of integration is bdd by

$x=0$ and $y=x$. The limits of 'x' are $\underline{x=0}$, $x=\infty$

and those of 'y' are $y=x$, $y=\infty$. The region

is shown in the figure.

On changing the order of integration, we



We first integrate w.r.t x , along a horizontal strip RS , which extends from $x=0$ to $x=y$.

To cover whole region of integration, we then integrate w.r.t. ' y ' from $y=0$ to ∞ .

Hence,

$$I = \int_0^{\infty} \left[\int_0^y \frac{e^{-y}}{y} dx \right] dy = \int_0^{\infty} \frac{e^{-y}}{y} [x]_0^y dy$$

$$= \int_0^{\infty} e^{-y} dy = [-e^{-y}]_0^{\infty} = -(0-1) = 1.$$

Ex. 3 Evaluate $\int_0^{\infty} \left(\int_0^x x e^{-\frac{x^2}{y}} dy \right) dx$, by changing the order of integration.

Sol: The region of integration is bdd by the lines

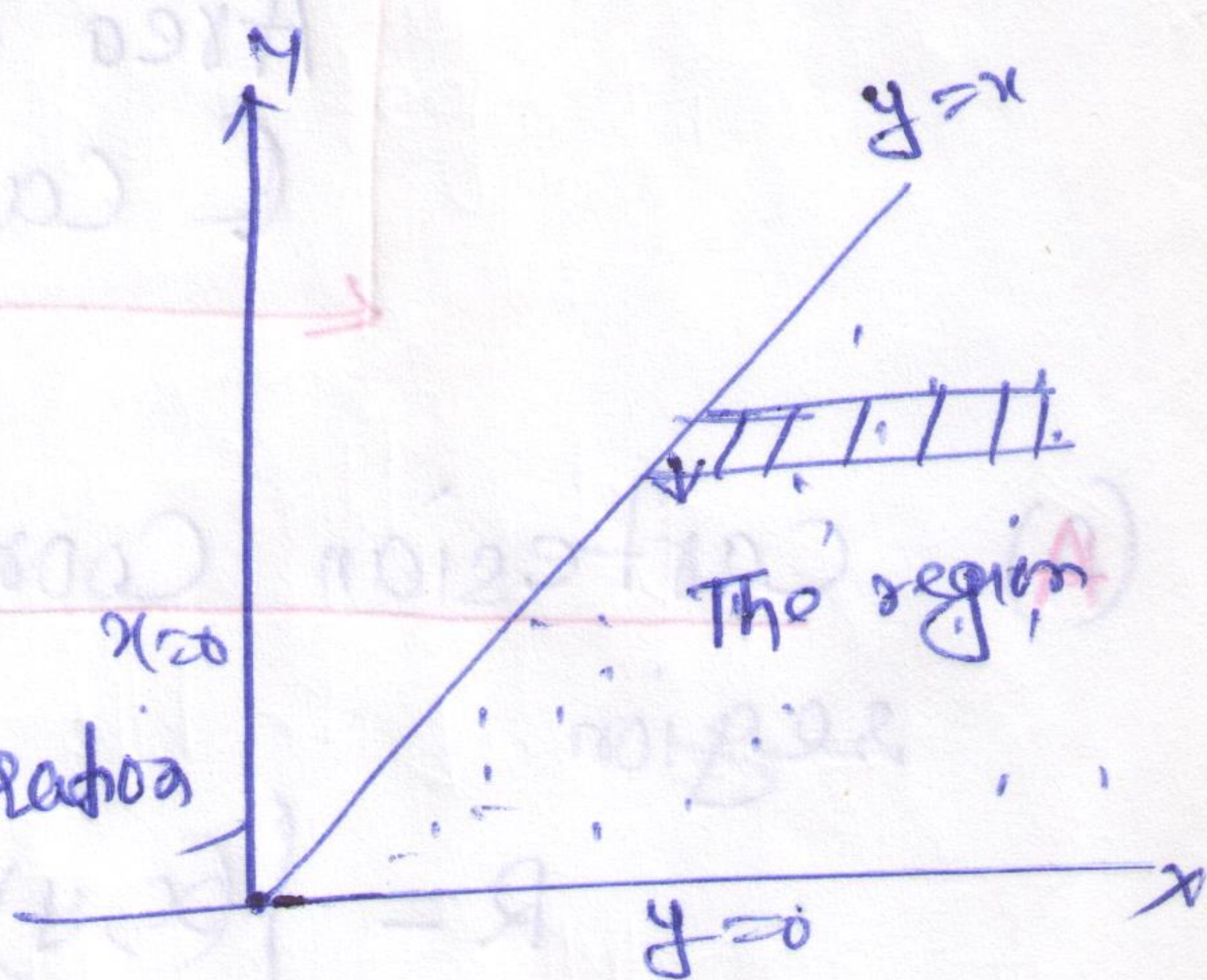
$$x=0, x=\infty, \underline{y=0}, \underline{y=x}.$$

The region of integration is as shown in the figures

On changing the order of integration

we first integrate w.r.t. (x) and then w.r.t. (y). Hence

$$\int_0^{\infty} \int_0^x x e^{-\frac{x^2}{y}} dy dx = \int_0^{\infty} \left[\int_y^{\infty} x e^{-\frac{x^2}{y}} dx \right] dy \quad (1)$$



First evaluate the inner integral.

Substituting $x^2 = t$, $2x dx = dt$.

When $x = y$, $t = y^2$ and $x = \infty$, $t = \infty$

$$\int_y^\infty x e^{-\frac{x^2}{y}} dx = \frac{1}{2} \int_{y^2}^\infty e^{-\frac{t}{y}} dt = \frac{1}{2} \left[\frac{e^{-\frac{t}{y}}}{-\frac{1}{y}} \right]_{y^2}^\infty$$

$$= \frac{1}{2} y e^{-y}.$$

Eq. (1) reduces to

$$\int_0^\infty \int_0^\infty x e^{-\frac{x^2}{y}} dy dx = \frac{1}{2} \int_0^\infty y e^{-y} dy \quad (\text{Gamma fun})$$

$$= \frac{1}{2} \int_0^\infty y^{2-1} e^{-y} dy = \frac{1}{2} \Gamma(2) = \frac{1}{2}.$$

($\because \Gamma(n+1) = n!$)

Area Enclosed By Plane Curve
(Cartesian and Polar Coordinates)

(A) Cartesian Coordinates: The area A of the region

$$R = \{(x, y) : a \leq x \leq b; f_1(x) \leq y \leq f_2(x)\}$$

is given by the double integral

$$A = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} dy \right) dx.$$

Similarly, the Area 'A' of the region (13)

$$R = \{ (x, y) : c \leq y \leq d, f_1(y) \leq x \leq f_2(y) \} \cup$$

is given by

$$A = \int_c^d \left(\int_{f_1(y)}^{f_2(y)} dx \right) dy.$$

(B) Polar Coordinates: The area A of the region $R = \{ (r, \theta) : \alpha \leq \theta \leq \beta, f_1(\theta) \leq r \leq f_2(\theta) \}$

is given by

$$A = \int_{\alpha}^{\beta} \left[\int_{f_1(\theta)}^{f_2(\theta)} r dr \right] d\theta. \quad \checkmark$$

Similarly, the area A of the region

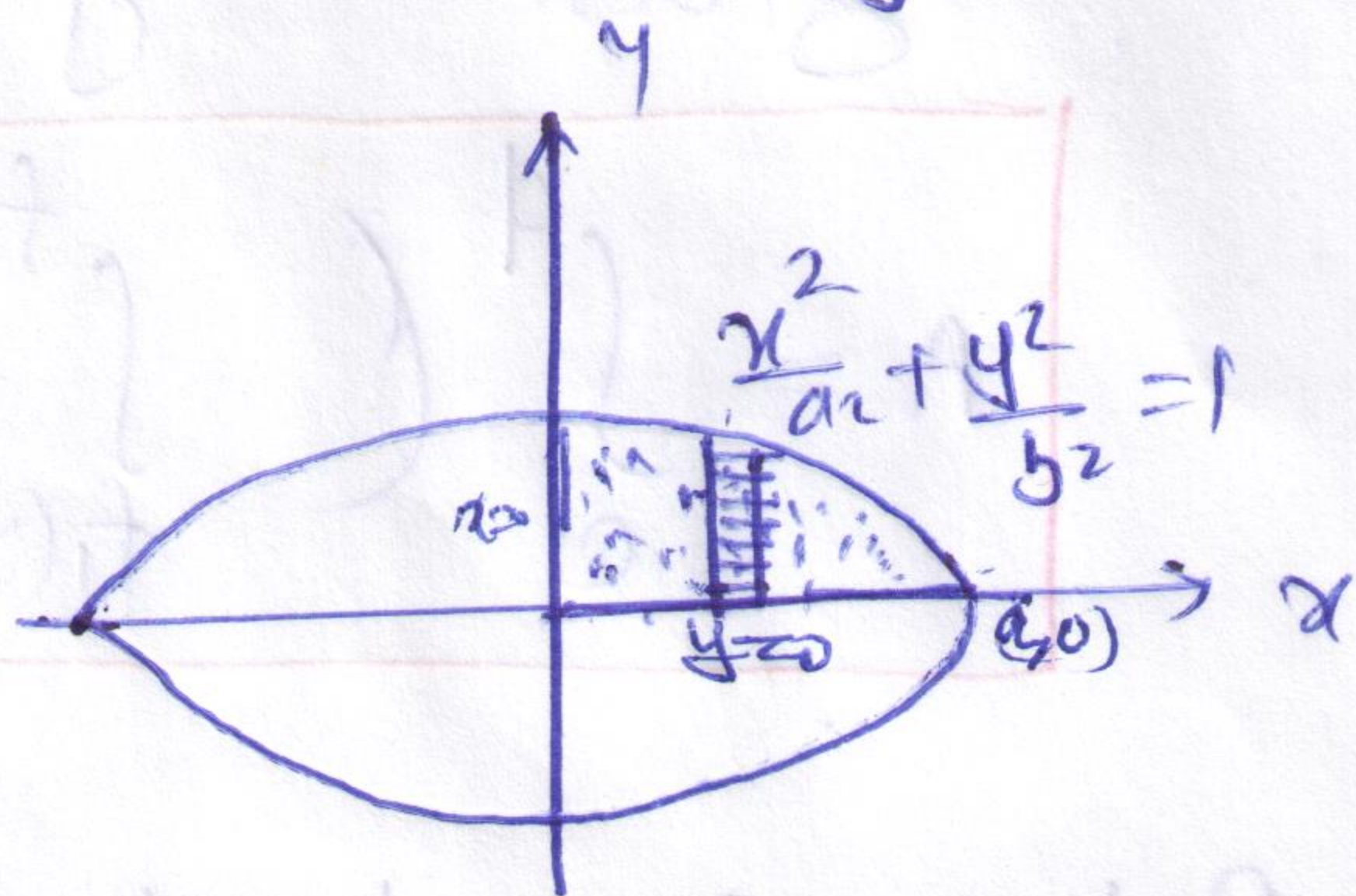
$$R = \{ (r, \theta) : r_1 \leq r \leq r_2, f_1(r) \leq \theta \leq f_2(r) \}$$

is given by

$$A = \int_{r_1}^{r_2} \left[\int_{f_1(r)}^{f_2(r)} d\theta \right] r dr.$$

Ex. 1: Find the area of a plate in the form of a ^{one} quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence, find the area enclosed by the given ellipse. (14)

Sol: From the figure, we note that the required area is bounded by



$$x=0, x=a, y=0 \text{ and } y = \frac{b}{a} \sqrt{a^2 - x^2}.$$

Thus,

$$A = \int_0^a \left(\int_0^{\frac{b}{a} \sqrt{a^2 - x^2}} dy \right) dx = \int_0^a \left[y \right]_0^{\frac{b}{a} \sqrt{a^2 - x^2}} dx$$

$$= \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx = \frac{b}{a} \left[\frac{x \sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{b}{a} \left[\frac{a^2}{2} \sin^{-1}(1) \right] = \frac{ab}{2} \sin^{-1} 1 = \frac{ab}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi ab}{4} \text{ Sq. units.}$$

Hence, the total area enclosed by the ellipse is 4 times the area enclosed by the plate in the form of one quadrant = πab Sq. units.

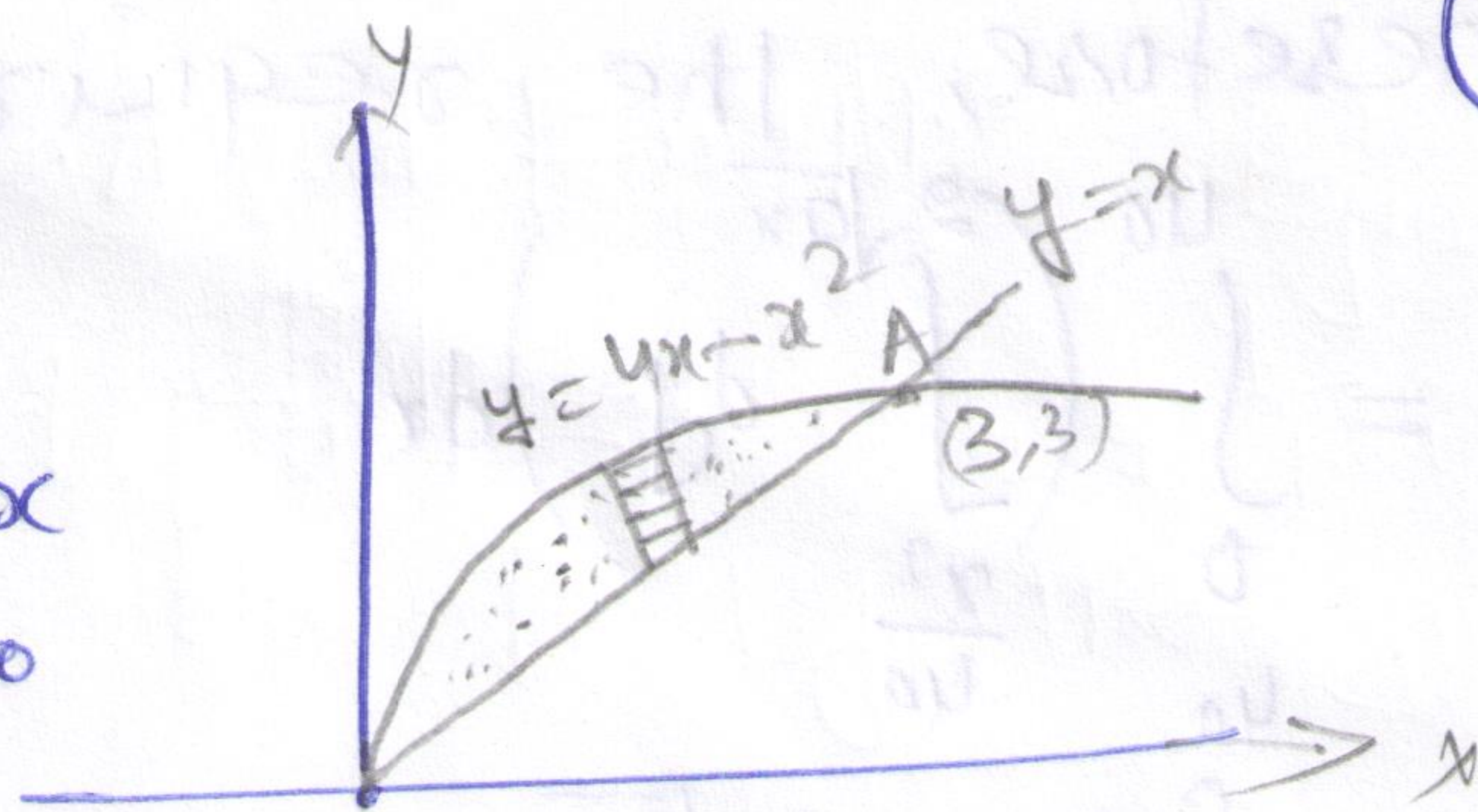
Ex 2. Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol: The parabola passing through the origin.

Solving $y = 4x - x^2$ and $y = x$

$$\text{i.e. } x = 4x - x^2 \Rightarrow x^2 - 3x = 0$$

$$x = 0 \text{ and } x = 3.$$



Thus, the curves $y = 4x - x^2$ and $y = x$ intersect at $x = 0, x = 3$.

For $0 < x < 3$, $4x - x^2$ is greater than x . Therefore, the region of integration is shown in the figure.

Thus, the required area lies between $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

$$\begin{aligned} A &= \int_0^3 \left(\int_x^{4x-x^2} dy \right) dx = \int_0^3 \left[y \right]_x^{4x-x^2} dx = \int_0^3 (3x - x^2) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}. \end{aligned}$$

Ex. 3 Find the area lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol: Solving the eqs. $y^2 = 4ax$, $x^2 = 4ay$, we get $(0,0)$ and $(4a,4a)$ as pts. of intersection.

The region of integration is shown in the figure.

Therefore, the required area is

$$A = \int_0^{4a} \left(\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy \right) dx$$

$$= \int_0^{4a} \left[y \right]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx = 2\sqrt{a} \int_0^{4a} \sqrt{x} dx - \frac{1}{4a} \int_0^{4a} x^2 dx$$

$$= 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a} = \frac{4}{3} \sqrt{a} (8a^{3/2}) - \frac{1}{12a} (64a^3)$$

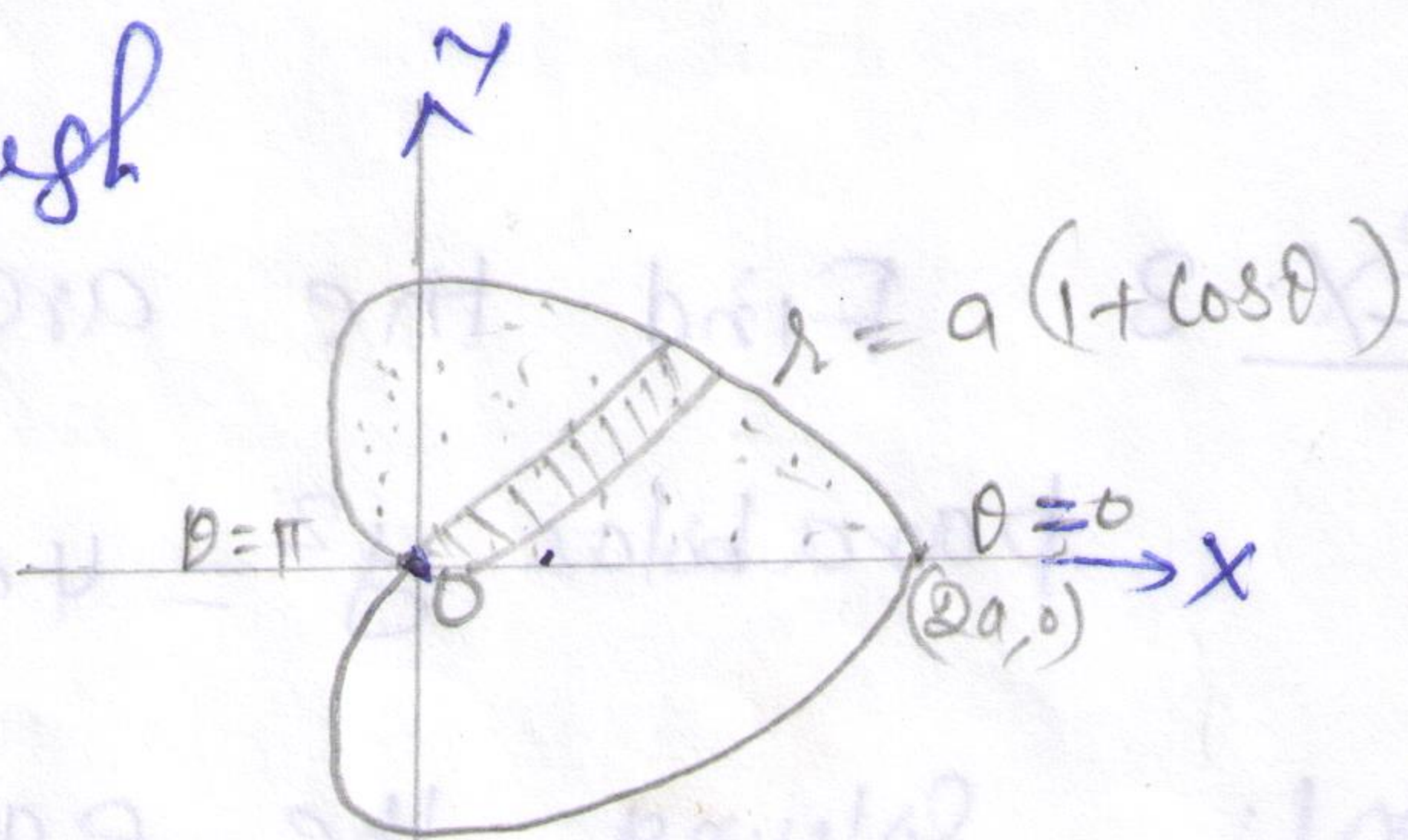
$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16}{3} a^2.$$

Ex. Find the area of the cardioid

$$r = a(1 + \cos \theta).$$

Sol. The curve passes through the origin and cuts the

x-axis at $x=2a$.



Clearly, θ varies from 0 to π

and x varies from 0 to $a(1 + \cos \theta)$ in

the upper half part of integration region.

The required area is given by

$$A = 2 \int_0^\pi \left[\int_0^{a(1+\cos\theta)} r \, dr \right] d\theta = 2 \int_0^\pi \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^\pi a^2 (1+\cos\theta)^2 d\theta = 4a^2 \int_0^\pi \cos^4 \frac{\theta}{2} d\theta \quad \left(\because 1+\cos\theta = 2\cos^2 \frac{\theta}{2} \right)$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi$$

$$\theta = 2\phi \\ d\theta = 2d\phi$$

$$= 8a^2 \cdot \frac{3}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}$$

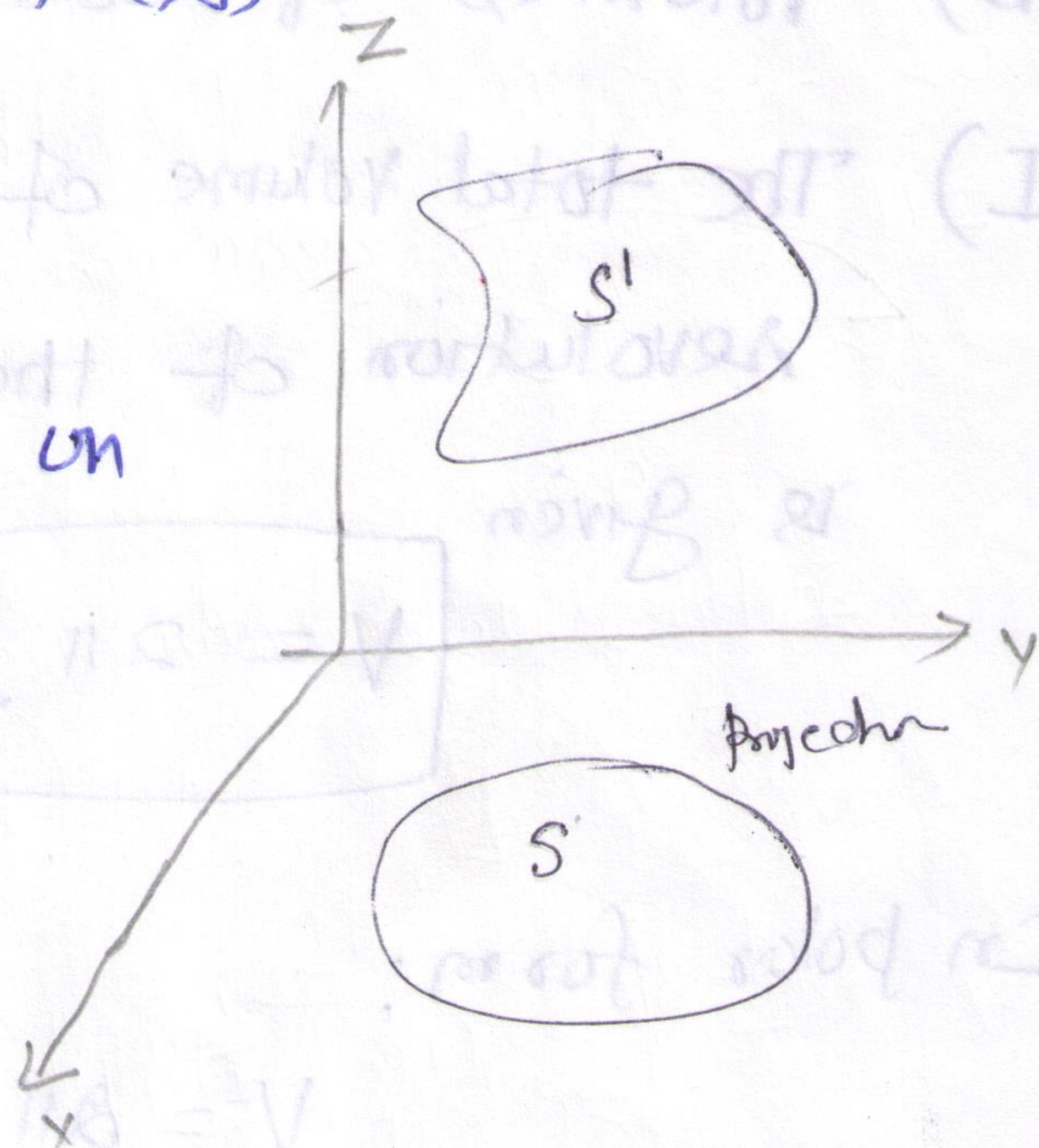
Volume And Surface Area As Double Integrals

(A) Volume as a Double Integral :-

Consider a surface $z = f(x, y)$.

Let the region S be the orthogonal projection of the portion S' of $z = f(x, y)$ on xy -plane.

Divide S into elementary rectangles of area $\delta x \delta y$ by drawing lines parallel to x - and y -axis.



(a) Volume of the Solids :-

Volume is given by

$$V = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dy dx.$$

This expression for V is called a repeated integral. If $z = f(x, y)$, then

$$\text{Volume } V = \iiint z dx dy$$

In polar coordinates:

$$V = \iint_S f(x \cos \theta, x \sin \theta) x dx d\theta.$$

(B) Volumes of Solids of Revolution:

(I) The total volume of the solid formed by the revolution of the area R about the x -axis is given

$$V = 2\pi \iint_R y dy dx \quad \text{--- (1)}$$

In polar form:

$$V = 2\pi \iint_R x \sin \theta (x dx d\theta) = 2\pi \iint_R x^2 \sin \theta dx d\theta. \quad \text{--- (2)}$$

(II) Similarly, the volume V of the area R revolved about the y -axis is given by

$$V = 2\pi \iint_R x dx dy \quad \text{--- (3)}$$

changing to polar coordinates,

$$V = 2\pi \iint_R r \cos \theta \, r \, dr \, d\theta = 2\pi \iint_R r^2 \cos \theta \, dr \, d\theta.$$

(c) Surface Area as a Double Integrals

Let $z = \psi(x, y)$ be a surface bounded by a curve C . Let the projection of C on xy -plane be bounded by Γ and let D be the domain on the xy -plane bounded by Γ .

Then, the area of the surface S is given by

$$S = \iint_D \sqrt{1 + (z_x)^2 + (z_y)^2} \, dx \, dy.$$

Ex. 1: Find the volume of sphere $x^2 + y^2 + z^2 = a^2$ using polar coordinates.

Sol: ~~The solid under consideration is bounded~~
consider a section by the plane $z=0$, which becomes

$$x^2 + y^2 = a^2, \quad z = \sqrt{a^2 - r^2}$$

Required Volume $V = 2 \cdot \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta$

$$= \int_0^{2\pi} \int_0^a (a^2 - r^2)^{1/2} (2r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{(a^2 - r^2)^{3/2}}{3/2} \right]_0^a d\theta = \frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}.$$

Ex. 2 Find the volume of the ellipsoid

(20)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol: Due to symmetry, the volume of the given ellipsoid is eight times the volume of the portion of the ellipsoid in the first Octant. For the positive Octant, the given equation yields

$$z = c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \text{ The region in this}$$

Octant is bounded by

$$x=0, x=a, y=0, y=b\sqrt{1-\frac{x^2}{a^2}}.$$

Hence, the required volume is given by

$$V = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} z \, dy \, dx$$

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx.$$

$$= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c \left[\left(1 - \frac{x^2}{a^2}\right) - \frac{y^2}{b^2} \right]^{\frac{1}{2}} \, dy \, dx$$

Substituting $\frac{y}{b} = \sqrt{1 - \frac{x^2}{a^2}} \sin \theta$, $dy = b \sqrt{1 - \frac{x^2}{a^2}} \cos \theta \, d\theta$

$$V = 8 \int_0^a \int_0^{\pi/2} c \left[\left(1 - \frac{x^2}{a^2}\right) - \left(1 - \frac{x^2}{a^2}\right) \sin^2 \theta \right]^{\frac{1}{2}} \times b \sqrt{1 - \frac{x^2}{a^2}} \cos \theta \, d\theta \, dx$$

$$= 8bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx \int_0^{\pi/2} \cos^2 \theta \, d\theta = 8bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx \cdot \frac{1}{2} \frac{\pi}{2}$$