Function of several Variables

Definition: A real valued function of 'n' variables is defined as

$$Z = f(x_1, x_2, \dots x_n),$$

$$x_1, x_2, \dots x_n \in \mathbb{R}^n, \quad Z \in \mathbb{R}$$

$x_1, x_2, \dots x_n$ are independent variables and z is the dependent variable.

Mathematically

$$f: \mathbb{R}^n \to \mathbb{R}$$

Function of two Variables:

$$Z = f(x, y)$$

The set of points (x, y) in the x-y plane for which $f(x,y)$ is defined is called domain of definition of the

function and it is denoted by D.
The collection of corresponding values of Z is called the range of the function f.

<u>Exc:</u> $Z = \sqrt{1-x^2-y^2}$

Domain: $1-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 1$

ie $D = \{(x,y) \mid x^2+y^2 \leq 1\}$

Range: set of all positive numbers between 0 and 1. including 0 & 1 also.

<u>Limit:</u> A function $f(x,y)$ approaches the limit l as (x,y) approaches (x_0, y_0) if $\forall \epsilon > 0$ $\exists$ a corresponding number $\delta > 0$ such that $\forall (x,y)$ in the domain D

$|f(x,y) - l| < \epsilon$ whenever

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

or whenever

$$0 < |x - x_0| < \delta$$

$$0 < |y - y_0| < \delta$$

$$\therefore \lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$$

<u>Exc:</u> Show that $\lim_{(x,y)\to(2,1)} 3x + 4y = 10$ using the ϵ-δ definition.

<u>Solution:</u> $f(x,y) = 3x + 4y$

$$|f(x,y) - 10| = |3x + 4y - 10|$$

$$= |3(x-2) + 4(y-1)|$$

$$\leq 3|x-2| + 4|y-1|$$

If we take

$0 < |x-2| < \delta$ and $0 < |y-1| < \delta$

We get

$$|f(x,y) - 10| < 7\delta < \epsilon$$

Which is satisfied when $\delta < \epsilon/7$

$\therefore$ $|f(x,y) - 10| < \epsilon$ whenever

$0 < |x-1| < \delta$

$0 < |y-1| < \delta$

$$\Rightarrow \lim_{(x,y)\to(3,2)} f(x,y) = 10$$

Exc: $\lim_{(x,y)\to(1,1)} (x^2 + 2y) = 3$

<u>Remark:</u> Since $(x,y) \to (x_0,y_0)$ in the two dimensional plane, there are infinite number of paths joining (x,y) to (x_0, y_0). Since the limit if exists is unique, the limit should be same along all the paths. Thus the limit cannot be obtained by approaching the point $P(x_0, y_0)$ along a particular path and finding the limit of $f(x,y)$. If the limit depends on a path, then the limit does not exist.

Exc: $f(x,y) = \dfrac{xy}{x^2+y^2}$

Find the limit when $(x,y) \to (0,0)$

Solution: Along $y = mx$

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

Limit depends on path.

$\therefore$ Limit does not exist.

Continuity: A function $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if

(i) $f(x,y)$ is defined at the point (x_0, y_0)

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Exc:

$$f(x,y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Change of coordinate system

$x = r\cos\theta$

$y = r\sin\theta$

Note that $r = \sqrt{x^2+y^2} \neq 0$

since $(x,y) \neq (0,0)$

Consider:

$$\left| f(x,y) - f(0,0) \right| = \left| \frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2(\cos^2\theta + \sin^2\theta)} \right|$$

$$= \left| 2r^2\cos^4\theta + 3r^2\sin^4\theta \right|$$

$$\leq 2r^2 + 3r^2$$

$$= 5r^2$$

$$< 5\delta^2$$

$$< \epsilon$$

$\Rightarrow$ choose $\delta < \sqrt{\epsilon/5}$

$\Rightarrow$ $\left| f(x,y) - f(0,0) \right| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\Rightarrow$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

$\Rightarrow$ Hence $f(x,y)$ is continuous at $(0,0)$.

Exc: $f(x,y) = \dfrac{xy}{x^2+y^2}$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{y\to 0}(0) = 0 \quad -(1)$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{x\to 0}(0) = 0 \quad -(2)$$

(1) and (2) are called repeated Limits.

Thus repeated limits exist and are equal.

But the simultaneous limit does not exist. Consider along the path $y = mx$

$$\lim_{x\to 0}\frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$ depends on the path

Thus Limit does not exist.

Repeated Limits

$$\lim_{y \to b} f(x,y) = \phi(x) \text{ exists}$$

and $$\lim_{x \to a} \phi(x) = l \text{ (exists)}$$

$$\therefore \lim_{x \to a} \left(\lim_{y \to b} f(x,y) \right) = l$$

& $$\lim_{y \to b} \left(\lim_{x \to a} f(x,y) \right) = l'$$

These two limits may or may not be equal.

Partial Derivative

Definition: The ordinary derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constant is called the partial derivative of the function with respect to that variable.

Let $z = f(x, y)$, $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0)$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0)$$

$$= \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Exc: $f(x,y) = ye^{-x}$: find f_x & f_y at (x,y)

$$f_x(x,y) = -ye^{-x}$$

By definition

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \frac{e^{-\Delta x} - 1}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \left[\frac{1 - \Delta x + \frac{\Delta x^2}{2} \cdots - 1}{\Delta x} \right]$$

$$= -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y+\Delta y)e^{-x} - ye^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x}$$

$$= e^{-x}$$

Exc: $f(x,y) = \sin(2x+3y)$

$f_x(x,y) = 2\cos(2x+3y)$

$f_y(x,y) = 3\cos(2x+3y)$

Relationship between continuity and the existence of partial derivatives

A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

Example: Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\dfrac{1}{x+y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives do not exist at $(0,0)$.

Solution:

$$|f(x,y) - f(0,0)|$$

$$= \left| (x+y) \sin\left(\frac{1}{x+y}\right) \right|$$

$$\leq |x+y|$$

$$\left.\begin{aligned} &\leq |x| + |y| \\ &\leq \sqrt{2} \cdot \sqrt{x^2+y^2} \end{aligned}\right\} \circledast$$

$$< \epsilon$$

choose $\delta < \epsilon/2$, then

$|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

Hence the function is Continuous at $(0,0)$.

$$\left(|x| - |y|\right)^2 \geq 0$$

$$\Rightarrow \quad x^2 + y^2 \geq 2|x|\,|y|$$

$$\Rightarrow \quad 2(x^2+y^2) \geq x^2 + y^2 + 2|x|\,|y|$$

$$\Rightarrow \quad 2(x^2+y^2) \geq \left(|x| + |y|\right)^2$$

$$\Rightarrow \quad |x| + |y| \leq \sqrt{2}\,\sqrt{x^2+y^2}$$

Now consider $x_0 = 0, \; y_0 = 0$

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \sin\left(\frac{1}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

$\Rightarrow f_x(0,0)$ does not exist. Also $\left(f_y(0,0)\right)$ does not exist

Example: Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$

Solution: Choose the path $y = mx$

The Limit

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + 2m^2x^2}$$

$$= \frac{m}{1+2m^2}$$

depends on the path.

Hence the function is not continuous at $(0,0)$.

Now consider

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0-0}{\Delta x}$$

$$= 0$$

$$= f_x(0,0)$$

$$\lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y}$$

$$= 0$$

$$= f_y(0,0)$$

$\Rightarrow$ The partial derivatives f_x and f_y at $(0,0)$.

Theorem: (Sufficient Condition for Continuity at (x_0, y_0)]

→ One of the partial derivative exist and is bounded in the neighbourhood of (x_0, y_0) and the other exist at (x_0, y_0)

Partial derivatives of Higher order

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xx} \qquad f_{yx} \qquad f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

are called second order partial derivatives of f.

→ The derivatives f_{xy} and f_{yx} are called mixed derivatives.

→ If f_{xy} and f_{yx} are continuous in open domain $\mathcal{D}$, then at any point $(x, y) \in \mathcal{D}$

$$f_{xy} = f_{yx}$$

Example: Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \cdot y^3}{\Delta x + y^2} \cdot \frac{1}{\Delta x}$$

$$= y$$

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{x \cdot \Delta y^3}{x + \Delta y^2} \cdot \frac{1}{\Delta y}$$

$$= 0$$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x}$$

$$= 0$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y - 0}{\Delta y}$$

$$= 1$$

$f_{yx}(0,0) = 1$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$

f_{xy} and f_{yx} are not continuous at $(0,0)$.

Homogeneous Function: An expression in (x, y) is homogeneous of order n if it can be expressed as $x^n f\left(\frac{y}{x}\right)$

Example:

1. $f(x,y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$

$$= x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + \cdots + a_n\left(\frac{y}{x}\right)^n\right]$$

$$= x^n g\left(\frac{y}{x}\right)$$

$\Rightarrow$ $f(x,y)$ is a homogeneous function of ~~degree~~ order n.

2. $f(x,y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x}$

$$= \frac{\sqrt{x}}{x}\left[\frac{\sqrt{\frac{y}{x}}+1}{\frac{y}{x}+1}\right]$$

$$= x^{-1/2}\, g(y/x)$$

$f(x,y)$ is a homogeneous function of order $-1/2$.

<u>Alternative Definition:</u> A function $f(x, y)$ is said to be homogeneous function of degree n if it Satisfies

$$f(tx, ty) = t^{n} f(x, y)$$

Euler's Theorem on Homogeneous Functions:

If $z = f(x, y)$ is a homogeneous function of x & y of order n, then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \forall\, x, y \in \mathcal{D}$$

$\mathcal{D}$ is the domain of the function $f(x,y)$.

Proof: Given

$$z = f(x, y)$$

$$= x^n g\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = n x^{n-1} g\left(\frac{y}{x}\right) + x^n \left(\frac{-y}{x^2}\right) g'\left(\frac{y}{x}\right)$$

$$= n x^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} y\, g'\left(\frac{y}{x}\right). \quad -(1)$$

$$\frac{\partial z}{\partial y} = x^n g'\left(\frac{y}{x}\right) \cdot \frac{1}{x} \quad -(2)$$

from 1 and 2

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right)$$

$$= nz$$

Theorem: If $z = f(x,y)$ is a homogeneous function of x and y of degree n, and second order partial derivatives are continuous then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Proof: As $z = f(x,y)$ is a homogeneous function of x and y of degree n,

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{(Euler's Theorem)} \quad \text{— (1)}$$

Differentiating (1) wrt x and y

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x} \quad \text{--- (2)}$$

$$x\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y} \quad \text{--- (3)}$$

$x\dfrac{\partial^2 z}{\partial y \partial x}$

$x \cdot (2) + y \cdot (3)$, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x \partial y}$$

$$+ xy\frac{\partial^2 z}{\partial y \partial x} + y\frac{\partial z}{\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

$$= n\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

$$= n^2 z - nz = n(n-1)z$$

$\left(\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}\right.$ $\because$ of continuity of second order derivatives$\left.\right)$

Exc: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$; $x \neq y$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

<u>Solution:</u> Let $z = \tan u$

$$= \frac{x^3+y^3}{x-y}$$

$$= x^2\left(\frac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}\right)$$

clearly z is a homogeneous function of degree 2.

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

subsitute $z = \tan u$ gives

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u$$

Exc: If $u = z e^{ax+by}$, where z is a homogeneous function in x and y of degree n.

Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (ax + by + n)\,u$$

Solution: Since z is a homogeneous function of degree n, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{(Euler's theorem)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{\partial z}{\partial x} e^{ax+by} + z e^{ax+by} . a\right] + y\left[\frac{\partial z}{\partial y} e^{ax+by} + z e^{ax+by} . b\right]$$

$$= e^{ax+by}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) + z\left[ax + by\right]e^{ax+by}$$

$$= \left[nz + z(ax+by)\right]e^{ax+by} = (ax + by + n)u$$

Exc: Let $Z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, where f and g are continuous and two times differentiable functions. Then evaluate

$$x^2 \frac{\partial^2 Z}{\partial x^2} + 2xy \frac{\partial^2 Z}{\partial x \partial y} + y^2 \frac{\partial^2 Z}{\partial y^2}$$

Solution: Let $Z = u_1 + u_2$, where

$u_1 = xy\, f(y/x) \rightarrow$ Homo. func of degree 2

$u_2 = g(y/x) \rightarrow$ Homo. function of degree 0.

$\therefore$ By Euler's theorem

$$x^2 \frac{\partial^2 u_1}{\partial x} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 2(2-1)u_1 \quad \text{—(1)}$$

$$+ \quad x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0 \quad \text{—(2)}$$

Adding (1) + (2)

$$x^2\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_2}{\partial x^2}\right)+2xy\left(\frac{\partial^2 u_1}{\partial x\partial y}+\frac{\partial^2 u_2}{\partial x\partial y}\right)$$

$$+y^2\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}\right)=2u_1$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$$

$$=2xy\,f(y/x)$$

Exc: If $Z = y + f(x/y)$, where f is continuous and differentiable function. Find

$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}$ Ans: y

Total Derivative

One variable:

Def 1: A function $y = f(x)$ is differentiable at a point (x, y) if

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists. The value of the above limit is called the derivative of x.

Def 2: A function $y = f(x)$ is said to be differentiable at a point (x, y) if

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= a\,\Delta x + \epsilon\,\Delta x,$$

Where a is independent of Δx

and $\lim_{\Delta x \to 0} \epsilon = 0$

The value of a is the derivative of f at x.

Remark: Def ① and ② are equivalent as

$$f(x+\Delta x) - f(x) = a\,\Delta x + \epsilon\,\Delta x$$

$$\Leftrightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = a + \epsilon$$

$$\Leftrightarrow \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = a$$

$(\epsilon \to 0$ as $\Delta x \to 0)$

Differential: The differential of the dependent variable y, written as dy, is defined to be

$$dy = f'(x)\,\Delta x$$

or $dy = f'(x)\,dx$; where $y = f(x)$

or $df = f'(x)\,dx$

Two Variable: The function $z = f(x,y)$ is said to be differentiable at the point (x,y) if at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

Where a and b are independent of Δx, Δy and ϵ_1 and ϵ_2 are functions of Δx and Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \quad \text{and} \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

The Linear function of Δx and Δy $a\Delta x + b\Delta y$ is called the total differential of Z

at the point (x, y) and is

denoted by dz.

$$dz = a\,\Delta x + b\,\Delta y$$
$$= a\,dx + b\,dy$$

If Δx and Δy are ~~dif~~ sufficiently

small, dz gives a close

approximation to Δz.

<u>Example:</u> Show that

$$z = x^2 + xy + xy^2$$

is differentiable and write down

its total derivative.

Solution:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= (x+\Delta x)^2 + (x+\Delta x)(y+\Delta y)$$

$$+ (x+\Delta x)(y+\Delta y)^2 - x^2 - xy$$

$$- xy^2$$

$$= \Delta x\,(2x+y+y^2)$$

$$+ \Delta y\,(x + 2xy)$$

$$+ (\Delta x + \Delta y\,(1+2y))\,\Delta x$$

$$+ (x\Delta y + \Delta x\,\Delta y)\,\Delta y$$

Hence the function is differentiable

$$\epsilon_1 = \Delta x + \Delta y\,(1+2y) \to 0$$

$$\epsilon_2 = x\Delta y + \Delta x\,\Delta y \to 0 \quad \text{as} \quad \begin{array}{l} \Delta x \to 0. \\ \Delta y \to 0 \end{array}$$

Total differentiable

$$dz = (2x+y+y^2)\,dx + (x+2xy)\,dy$$

Composite functions:

Consider $z = f(x, y)$ — ①

and let

$$\left.\begin{aligned} x &= u(t) \\ y &= \psi(t) \end{aligned}\right\} \text{—②} \quad \text{or} \quad \left.\begin{aligned} x &= \phi(u,v) \\ y &= \psi(u,v) \end{aligned}\right\} \text{—③}$$

The Equations ① and ② are said to define z as composite function of t or $(u \,\&\, v)$.

Differentiation of Composite functions

(Chain Rule)

Let $z = f(x, y)$ possess continuous partial derivatives (or differentiable)

and let $x = u(t)$

$y = \psi(t)$

possess continuous derivatives

Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

→ If $x = \phi(u, v)$
$y = \psi(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

& $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Exc $Z = xy$, $x = \cos t$, $y = \sin t$

Find $\frac{dz}{dt}$

Solution:
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= y(-\sin t) + x(\cos t)$$
$$= -\sin^2 t + \cos^2 t$$
$$= \cos 2t$$

Exc: Let z is a function of x & y. Prove that if

$$x = e^{u} + e^{-v}$$

$$y = e^{-u} + e^{v},$$ then

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^{u} + \frac{\partial z}{\partial y}\left(-e^{-u}\right)$$

$$= \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} \quad \text{--- (1)}$$

Similarly $$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v} \quad \text{--- (2)}$$

(1) − (2)

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(e^{u} + e^{-v}\right) + \frac{\partial z}{\partial y}\left(-e^{-u} - e^{v}\right)$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Derivative of a function defined implicitely

One variable: Let the function y of x be defined by

$$F(x, y) = 0$$

and let

$$Z \equiv F(x, y) = 0$$

$$\Rightarrow \quad \frac{dZ}{dx} \equiv \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y} \quad \text{if } \frac{\partial F}{\partial y} \neq 0$$

Two Independent variable:

$$F(x, y, z) = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0$$

and $\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}$ if $\frac{\partial F}{\partial z} \neq 0$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $x^2 + y^2 + z^2 - c = 0$

Solution: $\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$

$= \frac{-2x}{2z} = -x/z$

$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$

$= -\frac{2y}{2z} = \frac{-y}{z}$

OR $x^2 + y^2 + z^2 - c = 0$

$2x + 2z\frac{\partial z}{\partial x} = 0$

$\Rightarrow \frac{\partial z}{\partial x} = -x/z$

Similarly, $\frac{\partial z}{\partial y} = -y/z$

Jacobian

If u and v are functions of x & y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y and it is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \quad \text{or} \quad J\left(\frac{u,v}{x,y}\right)$$

Similarly Jacobian of u,v,w w.r.t x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Exc:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ and $\dfrac{\partial(r, \theta)}{\partial(x, y)}$

Solution:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \, , \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \, , \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Now
$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}$$

Result: If u and v are functions of x, y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Exc: If $x = uv$

$$y = \frac{u+v}{u-v}$$

Find $\dfrac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \dfrac{-2v}{(u-v)^2} & \dfrac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{4uv}{(u-v)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = (u-v)^2 / 4uv$$

<u>Result:</u> Let u, v are functions of r, s and r, s are functions of (x, y), then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

<u>Proof:</u> $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Exc: Find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

$$u = x^2 - y^2 \qquad x = r\cos\theta$$

$$v = 2xy \qquad y = r\sin\theta$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2+y^2)$$

$$= 4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= 4r^3$$

Result: If $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$

$\Rightarrow$ u, v, w are not independent.

Exc: $u = \dfrac{x+y}{1-xy}$

$v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$= 0$

$\Rightarrow$ u, v are not independent

i.e u and v are functionally related

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1}u$$

Jacobian of Implicit Functions

The variables x, y, u, v are connected by implicit functions

$$\left.\begin{aligned} f_1(x, y, u, v) &= 0 \\ f_2(x, y, u, v) &= 0 \end{aligned}\right\} \quad — (*)$$

u and v are implicit functions of x and y.

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

In general

$$f_1(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_n) = 0$$

$$\vdots$$

$$f_n(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_n) = 0$$

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots f_n) / \partial(x_1, \ldots x_n)}{\partial(f_1, f_2, \ldots f_n) / \partial(u_1, \ldots, u_n)}$$

Exc: $f_1 = x^2 + y^2 + u^2 - v^2 = 0$

$f_2 = uv + xy = 0$

$$\frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$$

$$= 2(x^2 - y^2)$$

$$\frac{\partial(f_1, f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

$$= 2(u^2 + v^2)$$

$$\frac{\partial(\cancel{x}\,u, \cancel{y}\,v)}{\partial(\cancel{u}\,x, \cancel{v}\,y)} = (-1)^2 \frac{\frac{\partial(f_1, f_2)}{\partial(x,y)}}{\frac{\partial(f_1, f_2)}{\partial(u,v)}} = \frac{2(x^2 - y^2)}{2(u^2 + v^2)}$$

$$= \frac{x^2 - y^2}{u^2 + v^2}$$

Taylor's Theorem for a function of two variables: Let a function $f(x, y)$ be defined in some domain $\mathcal{D}$ in $\mathbb{R}^2$ and have continuous partial derivatives up to $(n+1)$th order in some neighbourhood of a point $P(x_0, y_0)$ in $\mathcal{D}$. Then,

$$f(x_0+h,\ y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{\lfloor 2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \dots + \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n$$

Where the remainder is given by

$$R_n = \frac{1}{\lfloor n+1}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,\ y_0+\theta k),$$

$$0<\theta<1.$$

Exc: Obtain Taylor's formula $(n=2)$ for $f(x,y) = \cos(x+y)$ at $(0,0)$

Sol:

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$

$$+ \frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)$$

$$+ \frac{1}{\lfloor 3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y),$$

$$0<\theta<1$$

$f(0,0) = 1$

$f_x = -\sin(x+y) \Rightarrow f_x(0,0) = 0$

$f_y = -\sin(x+y) \Rightarrow f_y(0,0) = 0$

$f_{xx} = f_{yy} = f_{xy} = -\cos(x+y)$

At $(0,0)$, $f_{xx} = f_{yy} = f_{xy} = -1$

$f_{xxx} = f_{yyy} = f_{xxy} = f_{yyx}\Big|_{(\theta x, \theta y)}$

$= \sin(\theta x + \theta y)$

$\therefore$

$$f(x,y) = 1 + 0 + \frac{-1}{\lfloor 2}(x^2 + 2xy + y^2) + \frac{1}{\lfloor 3}(x^3 + 3x^2y + 3xy^2 + y^3). \sin(\theta x + \theta y)$$

$$= 1 - \frac{1}{\lfloor 2}(x+y)^2 + \frac{1}{\lfloor 3}(x+y)^3 \sin(\theta x + \theta y)$$

—

Maclaurin's Theorem

set $(x_0, y_0) = (0,0)$

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$

$$+ \frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0)$$

$$\vdots$$

$$+ \frac{1}{\underline{|n}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0,0)$$

$$+ R_n$$

$$R_n = \frac{1}{\underline{|n+1}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x, \theta y),$$

$$0 < \theta < 1$$

Exc: Find the second degree (Quadratic) Taylor's expansion to $f(x,y) = \dfrac{x-y}{x+y}$ about $(1,1)$

<u>solution:</u> $$f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\Rightarrow \quad f_x(1,1) = \frac{1}{2}$$

$$f_y = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\Rightarrow \quad f_y(1,1) = -1/2$$

$$f_{xx} = \frac{-4y}{(x+y)^3} \Rightarrow f_{xx}(1,1) = \frac{-1}{2}$$

$$f_{yy} = \frac{4x}{(x+y)^3} \Rightarrow f_{yy}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{2x-2y}{(x+y)^3} \Rightarrow f_{xy}(1,1) = 0$$

$f(x,y)$

$$f(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) + \frac{1}{2} f_{xx}(1,1)(x-1)^2 + f_{xy}(1,1)(x-1)(y-1) + \frac{1}{2} f_{yy}(1,1)(y-1)^2$$

$$= \frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 + \frac{1}{4}(y-1)^2$$

Maxima and Minima of a Function:

Definition: A function $z = f(x, y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighbourhood of (x_0, y_0) the function assumes a smaller value (or a larger value) than at the point itself. Such a maximum or minimum is often called relative (or local) maximum or minimum respectively.

Definition: For a given closed and bounded domain, a function may also attain its greatest value (or least value) on the boundary of the domain. The smallest and the largest values attained by a function over the entire domain including the boundary are called the absolute (or global) minimum and absolute (or global) maximum, respectively.

→ The point (x_0, y_0) is called Critical point (or stationary point) of $f(x,y)$ if $f_x(x_0,y_0) = 0$ and $f_y(x_0, y_0) = 0$

→ A critical point where the function has no minimum or maximum is called a saddle point.

→ Minimum and maximum values together are called extreme values.

Theorem: Necessary Condition for a function to have extremum:

Let $f(x, y)$ be continuous and have first order partial derivatives at a point $P(a, b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a, b) = 0$$

$$f_y(a, b) = 0$$

Or: If the point (a, b) is a relative extrema of the function $f(x, y)$ then (a, b) is also a critical point of $f(x, y)$.

Sufficient Conditions for a function to have Minima/ Maxima

For simplicity

$$r = f_{xx}(a, b)$$

$$s = f_{xy}(a, b)$$

$$t = f_{yy}(a, b)$$

Let a function $f(x, y)$ be continuous and have first and second order partial derivatives at a point $P(a, b)$.

If (a, b) is a critical point, then the point P is a point of

(i) Local maximum if $rt - s^2 > 0$ and $r < 0$

(ii) Local minimum if $rt - s^2 > 0$ and $r > 0$

(iii) Saddle point if $rt - s^2 < 0$

(iv) May be a local minimum, local maximum or saddle point if $rt - s^2 = 0$

Working Rule

1. Find critical points or Stationary points

$$f_x = 0 \ \& \ f_y = 0$$

2. For each critical point, evaluate

$$r = f_{xx}$$
$$s = f_{xy}$$
$$t = f_{yy}$$

3. Identification:

(i) If $rt - s^2 > 0$ & $r < 0 \rightarrow$ Maximum

(ii) If $rt - s^2 > 0$ & $r > 0 \rightarrow$ Minimum

(iii) If $rt - s^2 < 0 \rightarrow$ Saddle point

(iv) If $rt - s^2 = 0 \rightarrow$ Doubtful, needs further investigation

<u>Exc:</u> $f(x,y) = (4x^2+y^2)\,e^{-x^2-4y^2}$

<u>Solution:</u> $f_x(x,y) = e^{-x^2-4y^2}\left(8x - 2x(4x^2+y^2)\right)$

$$= e^{-x^2-4y^2}(8x - 8x^3 - 2xy^2)$$

$$= e^{-x^2-4y^2}(2x)(4-4x^2-y^2)$$

$$f_y(x,y) = e^{-x^2-4y^2}\left(2y - 8y(4x^2+y^2)\right)$$

$$= e^{-x^2-4y^2}(2y)(1-16x^2-4y^2)$$

<u>critical points</u> $f_x = 0$ & $f_y = 0$

(i) $x=0,\ y=0$

(ii) $x=0,\ y\neq 0,\ 1-4y^2=0 \Rightarrow y = \pm\frac{1}{2}$

(iii) $x\neq 0,\ y=0 \Rightarrow 4-4x^2=0 \Rightarrow x=\pm 1$

(iv) $x\neq 0,\ y\neq 0 \Rightarrow \left.\begin{array}{l} 4x^2+y^2=4 \\ \& \ 4x^2+y^2=\frac{1}{4}\end{array}\right\}$ no solution

Hence, critical points are

$P_1 = (0,0)$, $P_2 = \left(0, \frac{1}{2}\right)$, $P_3 = \left(0, \frac{-1}{2}\right)$

$P_4 = (1,0)$, $P_5 = (-1,0)$

$$r = f_{xx} = 2e^{-x^2-4y^2}\left(4 - 20x^2 + 8x^4 - y^2 + 2x^2y^2\right]$$

$$t = f_{yy} = 2e^{-x^2-4y^2}\left(1 - 20y^2 - 16x^2 - 128x^2y^2 + 32y^4\right)$$

$$s = f_{xy} = e^{-x^2-4y^2} \cdot 4xy\left(-17 + 16x^2 + 4y^2\right)$$

<u>Identification:</u>

$P_1\ (0,0)$ $\quad r = 8,\ s = 0,\ t = 2$

$rt - s^2 = 16 > 0$ & $r > 0$

$\Rightarrow P_1$ is a local minima.

$P_2\left(0, \frac{1}{2}\right)$ & $P_3\left(0, \frac{-1}{2}\right)$

$$r = 2e^{-1}\left(4 - \frac{1}{4}\right) = \frac{15}{2e}$$

$$s = 0$$

$$t = \frac{-4}{e}$$

$$rt - s^2 = \frac{-30}{e^2} < 0$$

$\Rightarrow$ P_2 and P_3 are saddle points.

$P_4(1, 0)$ & $P_5(-1, 0)$

$$r = -16e^{-1}$$

$$s = 0$$

$$t = -30e^{-1}$$

$$rt - s^2 = \frac{480}{e^2} > 0, \quad r < 0$$

Hence P_4 & P_5 are the points of Local maximum.

Exc:

$$f(x,y) = 2(x^2-y^2) - x^4 + y^4$$

$$f_x = 4x - 4x^3 = 0$$

$$\Rightarrow \quad x(x^2-1) = 0 \Rightarrow x = 0, \pm 1$$

$$f_y = -4y + 4y^3 = 0$$

$$\Rightarrow \quad 4y(y^2-1) = 0 \Rightarrow y = 0, \; y = \pm 1$$

Hence, $(0,0)$, $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \pm 1)$ are critical points.

$$r = f_{xx} = 4 - 12x^2$$

$$s = f_{xy} = 0$$

$$t = f_{yy} = -4 + 12y^2$$

At point $(0,0)$

$$rt - s^2 = (4-12x^2)(-4+12y^2) - 0$$

$$= (4-0)(-4+0) = -16 < 0$$

$\Rightarrow$ $f(x,y)$ has neither maxima nor minima at $(0,0)$.

At $(0,1)$

$$rt - s^2 = (4-12x^2)(-4+12y^2)$$
$$= 4(-4+12) = 32 > 0$$

and $r = 4 - 12x^2 = 4 > 0$

$\Rightarrow$ $f(x,y)$ has minimum value at

$(0,1)$

$\therefore$ $f(0,1) = 2(0-1) - 0 + 1 = -1$

is the minimum value.

At $(0,-1)$:

$rt - s^2 > 0$ and $r > 0$

$f(x,y)$ has minimum value given by

$$f(0,-1) = -1$$

At point $(-1,0)$ and $(1,0)$

$rt - s^2 = 32 > 0$ and $r = -8 < 0$

$\Rightarrow$ At point $(-1,0)$ and $(1,0)$

$f(x,y)$ have maximum value

i.e $f(\pm 1, 0) = 1$

At $(\pm 1, \pm 1)$, $rt - s^2 = -64 < 0$

$\Rightarrow$ $f(x,y)$ has neither maximum nor minimum at $(\pm 1, \pm 1)$.

Exc: show that

$$f = (y-x)^4 + (x-2)^4$$ has a minimum at $(2,2)$

Solution: $f_x = -4(y-x)^3 + 4(x-2)^3$

$r = f_{xx} = 12(y-x)^2 + 12(x-2)^2$

$s = f_{xy} = -12(y-x)^2$

$f_y = 4(y-x)^3$

$t = f_{yy} = 12(y-x)^3$

At $(2,2)$

$r = s = t = 0$

$\therefore \quad rt - s^2 = 0$

$\Rightarrow$ Further investigation needed

$\therefore \quad f(2+h, 2+k) - f(2,2)$

$= (2+k-2-h)^4 + (2+h-2)^2 - f(2,2)$

$= (k-h)^4 + h^4 > 0 \quad \forall \; h, k$

$\Rightarrow f(x,y)$ has minimum at $(2,2)$

and value is $f(2,2) = 0$.

Lagrange's Method of undetermined Coefficients:

Find the maximum/minimum value of the function $u = f(x,y)$ — (1)
with the following constraint

$$\phi(x,y) = 0 \quad — (2)$$

Rule:

1. Construct the auxilary function

$$F(x,y,\lambda) = f(x,y) + \lambda\phi(x,y)$$

2. Write the necessary conditions

$$F_x = 0 \Rightarrow f_x + \lambda\phi_x(x,y) = 0$$

$$F_y = 0 \Rightarrow f_y + \lambda\phi_y(x,y) = 0$$

$$F_\lambda = 0 \Rightarrow \phi(x,y) = 0$$

3. Further Investigate to test the nature of points.

In general:

Find extremum of $f(x_1, x_2, \ldots, x_n)$ subject to the conditions

$$\phi_i(x_1, x_2, \ldots x_n) = 0, \quad i = 0, 1, \ldots K$$

1. Construct the Auxilary function

$$F(x_1, \ldots x_n, \lambda_1, \ldots \lambda_K) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{K} \lambda_i \phi_i(x_1, \ldots x_n)$$

2. Find stationary points

$$\frac{\partial F}{\partial x_i} = 0, \quad i = 1, 2, \ldots n$$

$$\frac{\partial F}{\partial \lambda_j} = 0; \quad j = 1, 2, \ldots K$$

3. Investigate further to test the nature of the points.

Exc: Find maximum/ minimum of the function x^2-y^2-2x in the region $x^2+y^2 \le 1$

Solution: Auxilary function

1. $F(x,y,\lambda) = (x^2-y^2-2x) + \lambda(x^2+y^2-1) = 0$

2. $F_x = 0 \Rightarrow 2x - 2 + 2\lambda x = 0$

$F_y = 0 \Rightarrow -2y + 2\lambda y = 0$

$\Rightarrow 2y(\lambda - 1) = 0$

$\Rightarrow y = 0, \ \lambda = 1$

$F_\lambda = 0 \Rightarrow x^2+y^2 = 1$

If $y=0$, $x^2+y^2 = 1 \Rightarrow x = \pm 1$

If $\lambda = 1$, $4x - 2 = 0 \Rightarrow x = 1/2$

$x = 1/2 \Rightarrow x^2+y^2 = 1 \Rightarrow y = \pm \frac{\sqrt{3}}{2}$

$\therefore$ Points are $(1,0)$, $(-1,0)$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$

Function value at critical points

(i) $(1,0)$: $f(x,y) = -1$

(ii) $(-1,0)$: $f(x,y) = \boxed{3} \rightarrow$ Max

(iii) $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$: $f(x,y) = \boxed{\frac{-3}{2}}$ Min

Exc: $f(x,y,z) = 2x + 3y + z$

subject to $x^2 + y^2 = 5$ and $x + z = 1$

Solution: 1. $F(x,y,z) = 2x + 3y + z + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1)$

2. $F_x = 2 + 2\lambda_1 x + \lambda_2 = 0$

$F_y = 3 + 2\lambda_1 y = 0$

$F_z = 1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$

$F_{\lambda_1} = x^2 + y^2 - 5 = 0$

$F_{\lambda_2} = x + z - 1 = 0$

$\lambda_2 = -1$

$\Rightarrow \quad 1 + 2\lambda_1 x = 0$

$\& \quad 3 + 2\lambda_1 y = 0$

$\Rightarrow \quad x = \frac{-1}{2\lambda_1} \; ; \; y = \frac{-3}{2\lambda_1}$

Subsitute in $x^2 + y^2 = 5$ to get λ_1

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$$

$$\Rightarrow \quad \lambda_1 = \pm \frac{1}{\sqrt{2}}$$

For $\lambda_1 = \frac{1}{\sqrt{2}}$; we get $x = -\frac{\sqrt{2}}{2}$

$y = -\frac{3\sqrt{2}}{2}$

$\therefore \quad Z = 1 - x = \frac{2+\sqrt{2}}{2}$ and

$$f(x, y, z) = -\sqrt{2} - \frac{9\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}$$

$$= 1 - 5\sqrt{2}$$

For $\lambda_1 = \frac{-1}{\sqrt{2}}$

$$x = \frac{\sqrt{2}}{2}, \quad y = \frac{3\sqrt{2}}{2}, \quad z = 1 - x$$
$$= \frac{2-\sqrt{2}}{2}$$

$$\Rightarrow f(x, y, z) = \sqrt{2} + \frac{9\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2}$$
$$= 1 + 5\sqrt{2}$$