Rank of a matrix: A number r is said to be the rank of a matrix A if it possesses the following two properties:

1. There is atleast one square submatrix A of order r whose determinant is not equal to zero.

2. If the matrix A contains any square submatrix of order $r+1$, then the determinant of every square submatrix A of order $r+1$ should be zero.

→ In short the rank of the matrix is the order of any highest order non vanishing minor of the matrix.

Note: 1. $\rho(A) \le \min(m, n)$

$A_{m \times n}$ matrix

$\rho(A)$: rank of A.

2. $\rho(A) = n$, $A_{n \times n}$, $|A| \neq 0$

3. $\rho(A) \geq 1$, $A \neq 0$

4. $\rho(O) = 0$

5. $A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$|A| = 1 \neq 0$

$\rho(A) = 3$

6. $A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$

To find the rank of A, start with the highest order minor.

$\Rightarrow |A| = 0$

$$\Rightarrow \left|\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow |3| = 3 \neq 0$

$\Rightarrow \rho(A) = 1$

Note: This method involves a lot of computation work, since we have to evaluate several determinants.

Echelon Form of a Matrix:

(Row Echelon Form)

A matrix A is said to be in Echelon form if

1. Every rows of A which has all its enteries 0 occurs below every row which has a non-zero entry.
2. The first nonzero entry in each nonzero row is equal to 1.
3. The number of zeros before the first nonzero element in a row is less than the number of such zeros in the next row.

Note: Condition 2 is not required

Exc:
$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = \# \text{ of non zero rows of } A$$
$$= 2$$

Note: $\rho(A') = \rho(A)$

Exc: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3 \times 3}$

$$\rho(A) = 1$$

Elementary operations or Elementary transformations of a matrix

1. The interchange of any two rows ($R_i \leftrightarrow R_j$)

2. The multiplication of the elements of any row by any nonzero number

$$R_i \rightarrow a R_i$$

3. The addition to the elements of any other row the corresponding elements of other row multiplied by any ~~non~~ number

$$R_i \rightarrow R_i + bR_j$$

Elementary Matrices: A matrix obtained from a unit matrix by a single elementary transformation is called elementary matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: Elementary transformation do not change the rank of a matrix.

Row Reduced Echelon Form

Every matrix can be transformed to row reduced Echelon form by a finite number of elementary row operations.

Exc: Find rank

1. $$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{pmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{pmatrix} \qquad R_3 \to 2R_3 + 3R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

$\rho(A) = 3$

2. $$A = \begin{pmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \quad R_2 \to R_2 - 4R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix} \quad \begin{matrix} R_3 \to 5R_3 - 3R_2 \\ R_4 \to 5R_4 - R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 0 & 24 & -22 \\ 0 & 0 & 8 & -4 \end{pmatrix} \quad R_4 \to 3R_4 - R_3$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -8 & 14 \\ 0 & 0 & 24 & -22 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Which is in Echelon Form

$\rho(A) = 4$

3. $$A = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 4 & -1 \end{pmatrix}$$

Since $a_{11} = 0$

$R_1 \leftrightarrow R_2$ (to make first entry non zero)

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 4 & -1 \end{pmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & -4 \end{pmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$\rho(A) = 3$

4. $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$ $\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{array}$

$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{pmatrix}$ $R_2 \leftrightarrow R_4$

$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ $R_4 \to R_4 + R_3$

$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = 3$

Vectors: Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \dots, x_n$ be any n numbers, then the ordered n-tuple $X = (x_1, x_2, \dots, x_n)$ is called an n-vector.

- A vector may be written either as a row vector or a coloumn vector.
- $(x_1, x_2, \dots, x_n) = 0$ if

  $x_i = 0 \quad \forall\, i = 1, 2, \dots n$

  $0 = (0, 0, \dots, 0)$ is called 0 vector.

- Let $X = (x_1, x_2, \ldots, x_n)$

$Y = (y_1, y_2, \ldots, y_n)$

1. Equality $X = Y$ iff $x_i = y_i \; \forall i$

2. Addition $X + Y = (x_1 + y_1, \ldots, x_n + y_n)$

3. $kX = (kx_1, kx_2, \ldots, kx_n)$

(Multiplication by a scaler)

Properties:

1. $X + Y = Y + X$

2. $X + (Y + Z) = (X + Y) + Z$

3. $p(X + Y) = pX + pY$

4. $(p + q)X = pX + qX$

5. $p(qX) = (pq)X$

p, q are scalers

X, Y, Z are vectors.

Linearly Dependent vectors

A set of r vectors $x_1, x_2, \ldots x_r$ is said to be linearly dependent if there exist scalers $k_1, k_2, \ldots, k_r$ not all zero such that

$$k_1 x_1 + k_2 x_2 + \ldots + k_r x_r = 0$$

Linearly Independent vectors: A set of r vectors $x_1, x_2, \ldots x_r$ is said to be linearly independent if every relation of the type

$$k_1 x_1 + k_2 x_2 + \ldots + k_r x_r = 0$$

$$\Rightarrow k_1 = k_2 = \ldots = k_r = 0$$

Exc: $X_1 = (1, 2, 4)$

$X_2 = (3, 6, 12)$

$$k_1(1,2,4) + k_2(3,6,12) = (0,0,0)$$

$$\Rightarrow (k_1 + 3k_2,\ 2k_1 + 6k_2,\ 4k_1 + 12k_2) = (0,0,0)$$

$$\Rightarrow \quad k_1 + 3k_2 = 0$$

$$2k_1 + 6k_2 = 0$$

$$4k_1 + 12k_2 = 0$$

$$\Rightarrow \quad k_1 = -3k_2$$

if $k_2 = 1$, then $k_1 = -3$

$\Rightarrow$ X_1 and X_2 are linearly dependent.

2. $(1,0,0)$, $(0,0,1)$, $(0,1,0)$

consider

$$k_1(1,0,0)+k_2(0,0,1)+k_3(0,1,0) = (0,0,0)$$

$\Rightarrow \quad k_1 = k_2 = k_3 = 0$

$\Rightarrow$ $(1,0,0)$, $(0,0,1)$, $(0,1,0)$ are linearly independent.

Linear System of Equations

Homogeneous Linear system of Equations

Consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\} \quad -(1)$$

is a system of m homogeneous equations in n unknowns $x_1, x_2, \dots x_n$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ $\quad O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m\times 1}$

Thus, the system of eq 1 can be writen as

$$AX = 0 \quad - \quad (2)$$

A : Coefficient Matrix

Note: 1. obviously $x_1 = x_2 = \cdots = x_n = 0$ i.e $X = 0$ is a solution of 1. It is a trivial solution of 1.

2. If X_1 and X_2 are two solutions of 2, then $k_1 X_1 + k_2 X_2$, k_1, k_2 are scalers, is also a solution.

$$AX_1 = 0$$
$$AX_2 = 0$$
$$A(k_1 X_1 + k_2 X_2) = k_1 AX_1 + k_2 AX_2$$
$$= k_1 \cdot 0 + k_2 \cdot 0 = 0$$

3. The number of linearly independent solutions of m homogeneous linear equations in n variables $AX = 0$ is $n - r$, where r is the rank of the matrix A.

Case 1: If $r = n$, $n - r = 0$
only 0 is the solution.

Case 2: If $r < n$, then $n - r$ linearly independent solutions.

4. Any linear combination of these $n - r$ solutions will also be a solution of $AX = 0$.

Solve

$$x + y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{pmatrix} \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{pmatrix} \qquad R_3 \to R_3 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rho(A) = 3 =$ number of unknowns

$\Rightarrow$ $AX = 0$ does not possess any linearly independent solution.

$\Rightarrow$ ie $x = y = z = 0$ is the ~~only~~ trivial solution of $AX = 0$

Exc: Solve
$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) = 2 <$ # of unknowns $= 3$

$\therefore$ $n - r = 3 - 2 = 1$ Linearly independent sol.

$$x + 3y - 2z = 0$$
$$-7y + 8z = 0$$
$$\Rightarrow y = \frac{8}{7} z$$

$$x = \frac{-10}{7} z$$

Choose $z = c$, c is arbitrary constant

$$y = \frac{8}{7} c$$

$$x = \frac{-10}{7} c$$

$\left(\frac{-10}{7} c, \frac{8}{7} c, c\right)$ is the general solution

→ We usually take $c = 1$

$\therefore \left(\frac{-10}{7}, \frac{8}{7}, 1\right)$ is the general sol

→ $(-10, 8, 7)$ is the general sol.
(Multiply by 7)

Discuss the Values of K for the system of Equations:

$$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

$$A = \begin{bmatrix} 2 & 3K & 3K+4 \\ 1 & K+4 & 4K+2 \\ 1 & 2K+2 & 3K+4 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & K+4 & 4K+2 \\ 2 & 3K & 3K+4 \\ 1 & 2K+2 & 3K+4 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & K+4 & 4K+2 \\ 0 & K-8 & -5K \\ 0 & K-2 & -K+2 \end{pmatrix}$$

$$R_3 \to (K-8)R_3 - (k-2)R_2$$

$$A = \begin{pmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & 0 & (k-8)(-k+2)+5k(k-2) \end{pmatrix}$$

For the rank A to be less than 3

$$(k-8)(-k+2) + 5k(k-2) = 0$$

$$\Rightarrow \quad k = \pm 2$$

<u>Case 1:</u> If $k \neq \pm 2$

$$\rho(A) = 3$$

$\Rightarrow$ system of Equations has only one solution $x = y = z = 0$.

<u>Case 2:</u> If $k = 2$

$$A = \begin{pmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x + 6y + 10z = 0$$
$$-6y - 10z = 0$$

$\Rightarrow \quad y = \frac{-5}{3} z$

$\Rightarrow \quad x = 0$

solution is $\left(0, -\frac{5}{3}, 1\right)$

ie $(0, -5, 3)$

<u>Case 3:</u> $\quad k = -2$

$$A = \begin{pmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x + 2y - 6z = 0$$
$$-10y + 10z = 0$$

$\Rightarrow \quad y = z$

$\quad \& \; x = 4y = 4z$

$\therefore \; (4, 1, 1)$ is the solution.

<u>Consistency condition:</u> The system of equations $Ax = b$ is consistent i.e. possess a solution iff

$$\rho(A) = \rho(A|b)$$

Check the consistency of the system

$$2x + 6y + 11 = 0$$
$$6x + 20y - 6z + 3 = 0$$
$$6y - 8z + 1 = 0$$

$$A = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{pmatrix}, \quad b = \begin{pmatrix} -11 \\ -3 \\ -1 \end{pmatrix}$$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right)$$

$$R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right) \quad R_3 \to R_3 - 3R_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right)$$

$\rho(A) = 2$

$\rho(A|b) = 3$

$\rho(A) \neq \rho(A|b)$

$\Rightarrow$ system is inconsistent.

Solution of Inhomogeneous system of Equations $Ax=b$

Case 1: If $\rho(A) < \rho(A|b)$ (or $\rho(A) \neq \rho(A|b)$)

the system $Ax=b$ is inconsistent.

The system has no solution.

Case 2: $\rho(A) = \rho(A|b) = r$

consistent

$Ax=b$ has $n-r+1$ has Linearly Independent solutions.

(i) $\rho(A) = r = n$ = number of unknowns

unique solution

(ii) $\rho(A) = r < n$

Infinite solutions but

$n-r+1$ linearly independent solutions.

Exc: $2x + 6y + 11 = 0$

$$6x + 20y - 6z + 3 = 0$$

$$6y - 18z + 1 = 0$$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 6 & 2 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right) \quad R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{ccc|c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right)$$

$\rho(A) = 2$

but $\rho(A|b) = 3$

Inconsistent

$\Rightarrow$ no solution.

Exc:

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right) \quad R_3 \to R_3 - 3R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rho(A) = \rho(A|b) = 2$$

$n - r + 1 = 3 - 2 + 1 = 2$ Linearly indep. solution

$$x + y + z = 6$$
$$y + 2z = 8$$

$$\Rightarrow y = 8 - 2z$$

$$\& \quad x = z - 2$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} z-2 \\ -2z+8 \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} z + \begin{pmatrix} -2 \\ 8 \\ 0 \end{pmatrix}$$

Solution set

$$= \left\{ \begin{pmatrix} -2 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ +6 \\ 1 \end{pmatrix} \right\}$$

Exc:

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

$$(A|b) = \left(\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right) \quad \begin{array}{l} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 5 & -17 & -24 \end{array}\right) \quad R_3 \to 3R_3 - 5R_2$$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -76 & -152 \end{array}\right)$$

$\rho(A) = \rho(A|b) = 3$ (unique solution)

$$-76z = -152 \Rightarrow z = 2$$

$$3y + 5z = 16 \Rightarrow y = 2$$

$$2x - y + 3z = 8 \Rightarrow x = 2$$

Investigate for what values of λ and μ the simultaneous equations have

1. no solution
2. unique solution
3. Infinite number of solutions

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right) \quad R_3 \to R_2 - R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right)$$

for

(i) no solution

$\lambda = 3, \mu \neq 10$

(ii) unique solution

$\lambda \neq 3$

(iii) Infinite solution

$\lambda = 3$ and $\mu = 10$.

Eigen Value: If there is a vector $X \neq 0 \in R^n$ such that

$$AX = \lambda X \quad - (1)$$

for some scaler λ, then λ is called the Eigen value of A and X is the corresponding Eigen vector.

Note: Now

$$AX = \lambda X$$

$$AX - \lambda I X = 0$$

$$(A - \lambda I) X = 0 \quad - (2)$$

(2) is a homogeneous system of Equations

As $X \neq 0 \Rightarrow |A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

Which is a polynomial in λ of degree n. This polynomial is called characterstic polynomial of A.

→ $|A - \lambda I| = 0$ is called the characterstic Equation of A and the roots of this equation are called characterstic roots or Eigen Values of A.

<u>Note:</u> λ is an Eigen value of $A \iff \exists\ X \neq 0$ such that

$$AX = \lambda X$$

Result: 1. If X is a characterstic vector of a matrix A corresponding to the characterstic value λ, then kX is also a characterstic vector of A corresponding to the same characterstic value λ, here k is any scaler.

$$AX = \lambda X$$

$$A(kX) = k(AX)$$

$$= k(\lambda X)$$

$$= \lambda(kX)$$

$\Rightarrow$ kX is an Eigen vector.

Note 2. If X is an Eigen vector of a matrix A, then X can not correspond to more than one characterstic values of A.

Proof:

$$AX = \lambda_1 X$$

$$AX = \lambda_2 X$$

$$\lambda_1 X - \lambda_2 X = 0$$

$$(\lambda_1 - \lambda_2) X = 0$$

As $X \neq 0$

$\Rightarrow \lambda_1 - \lambda_2 = 0$

$\Rightarrow \lambda_1 = \lambda_2$

Exc: $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$$

$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$

$\Rightarrow \lambda = 1, 6$

Eigen vector: $\lambda_1 = 6$

$$(A - \lambda_1 I) X = 0$$

$$(A - 6I) X = 0$$

$$\Rightarrow \begin{pmatrix} 5-6 & 4 \\ 1 & 2-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 - 4x_2 = 0$$

$$x_1 = 4x_2$$

Take $x_2 = 1$

$x_1 = 4$

$X_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an Eigen vector

Corresponding to $\lambda_1 = 6$

Eigen vector Corresponding to $\lambda_2 = 1$

$$(A - \lambda_2 I)\, X = 0$$

$$(A - I)\, X = 0$$

$$\Rightarrow \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = 0$$

$$\Rightarrow \quad x_1 = -x_2$$

$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an Eigen vector Corresponding to $\lambda_2 = 1$.

Exc: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)^3 = 0$

$\Rightarrow \quad \lambda = 2, 2, 2$ are Eigen values.

$$(A - 2I)X = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x_2 = 0, \quad x_3 = 0$

$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigen vector corresponding to $\lambda = 2$.

Exc:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$\Rightarrow$ $\lambda = 2, 2, 2$ are Eigen values

$$(A - 2I)X = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_3 = 0$$

$\therefore$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are Eigen vectors

Corresponding to $\lambda = 2$

Exc:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$\Rightarrow$ $\lambda = 2, 2, 2$ are Eigen values

$$(A - 2I) X = 0$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\Rightarrow$ x_1, x_2, x_3 are independent

$\therefore$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are

Eigenvectors corresponding to

$\lambda = 2, 2, 2.$

Cayley Hamilton Theorem:

Every square matrix satisfies its own characterstic equation i.e for a square matrix A of order n,

$$|A-\lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + \dots + a_n]$$

the matrix equation

$$X^n + a_1 X^{n-1} + \dots + a_n I = 0$$

is satisfied by $X = A$, i.e,

$$A^n + a_1 A^{n-1} + \dots + a_n I = 0$$

Verify Cayley Hamilton Theorem

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

Find A^{-1} and A^{8}.

Solution: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 5 = 0$ (characterstic Equation)

To show $A^2 - 5I = 0$

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

Hence, the result.

A satisfies characterstic equation.

→ To find A^{-1}

$$A^2 - 5I = 0$$

Multiply by A^{-1}

$$A^{-1} \cdot A^2 - 5A^{-1}I = 0$$

$$\Rightarrow \quad A - 5A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}A$$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

To find A^8: Multiply by A^6 to $A^2 - 5I$

$$A^6 \cdot A^2 - 5I A^6 = 0$$

$$\Rightarrow \quad A^8 = 5A^6 = 5A^2 \cdot A^2 \cdot A^2$$

$$= 5(5I)(5I)(5I)$$

$$= 625I$$

Exc: Verify Cayley Hamilton Theorem.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}. \quad \text{Find } A^{-1} \text{ and } A^4.$$

Solution: $|A - \lambda I| = 0$

$$\Rightarrow \quad \lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0 \quad - (1)$$

To show (1) is satisfied by A

ie to show $A^3 - 11A^2 - 4A + I = 0$

$$A^2 = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 36 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix}$$

$$A^3 - 11A^2 - 4A + I = 0$$

To find A^{-1}

$$A^{-1} = -A^2 + 11A + 4I$$

$$= \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

To find A^4

$$A^3 = 11A^2 + 4A - I$$

$$A^4 = 11A^3 + 4A^2 - A$$

$$= \begin{pmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & -8957 \end{pmatrix}$$

$$B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$$

$$= A^5(A^3 - 11A^2 - 4A + I) + A(A^3 - 11A^2 - 4A + I) + A^2 + A + I$$

$$= A^2 + A + I$$

$$= \begin{pmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 61 & 77 \end{pmatrix}$$