1

Sequence: A sequence is a list of numbers written in a specific order. Terms of the sequence are denoted as follows.

a_1 : first term

a_2 : second term

a_3 : third term

$\vdots$

a_n : nth term

a_{n+1} : $(n+1)$th term

Notation: A sequence is denoted by $\langle a_n \rangle$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

(2)

Exc: Write down the first few terms of the following sequence

1. $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

$= \left\{\frac{2}{1^2}, \frac{3}{2^2}, \frac{4}{3^2}, \frac{5}{4^2}, \dots\right\}$

2. $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=1}^{\infty}$

$= \left\{\frac{+1}{2}, \frac{-1}{4}, \frac{+1}{8}, \frac{-1}{16}, \dots\right\}$

Mathematical Definition: Sequence is a function whose domain is a set of natural number and codomain is real.

Symbolically $a: \mathbb{N} \to \mathbb{R}$

③

$$a(n) = a_n$$

a_n is called the nth term of the sequence $\langle a_n \rangle$ or $\{a_n\}$.

<u>Range of a sequence:</u> The set of all distinct terms of a sequence is called its range.

<u>Note:</u> 1. The number of terms of a sequence is always infinite.

2. The range of a sequence may be a finite set.

e.g. $\{a_n\} = \{(-1)^n\}$

$$\langle a_n \rangle = \{-1, 1, -1, 1, -1, 1, \ldots\}$$

$$\text{Range} = \{-1, 1\} : \text{finite set}$$

(4)

Bounded sequence: A sequence $\langle a_n \rangle$ is said to be bdd (bounded) if $\exists$ two real numbers k and K such that

$$k \leq a_n \leq K \quad \forall\, n \in \mathbb{N}$$

or

choose $M = \max\{|k|, |K|\}$,

then $|a_n| \leq M \quad \forall\, n \in \mathbb{N}$

Note:

1. If $a_n \leq K \quad \forall\, n \in \mathbb{N}$

   $\langle a_n \rangle$ is bounded above

2. If $a_n \geq k \quad \forall\, n \in \mathbb{N}$

   $\langle a_n \rangle$ is bounded below.

5

Unbounded sequence: A sequence $\langle a_n \rangle$ is said to be unbounded if $\nexists$ real number M such that

$$|a_n| \leq M \quad \forall\, n \in \mathbb{N}.$$

Excercises: 1. $\langle a_n \rangle = \langle \frac{1}{n} \rangle$

$$0 < a_n \leq 1 \quad \forall\, n \in \mathbb{N}$$

: Bounded

2. $\langle a_n \rangle = \langle (-1)^n \rangle$

$$-1 \leq a_n \leq 1, \quad \forall\, n \in \mathbb{N}$$

Bounded

3. $\langle a_n \rangle = \langle 2^{n-1} \rangle$

$$a_n \geq 1 \quad \forall\, n \in \mathbb{N}$$

Not Bounded above (Bounded below)

4. $\langle a_n \rangle = \langle n \rangle$

$$a_n \geq 1 \quad \forall\, n \in \mathbb{N}$$

Bounded below

6

Convergence of sequence

1. If $\lim_{n\to\infty} a_n = l$, then

$\langle a_n \rangle$ converges to l.

• l is called Limit of a sequence.

2. If $\lim_{n\to\infty} a_n = \infty$ or $-\infty$, then

$\langle a_n \rangle$ is divergent.

Exc: (1.) $\langle a_n \rangle = \left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0 \text{ (finite)}$$

$\Rightarrow \left\langle \frac{1}{2^n} \right\rangle$ is Convergent.

(2.) $\langle a_n \rangle = \langle n^2 \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$$

$\langle a_n \rangle$ is divergent.

(7)

3. If a sequence $\langle a_n \rangle$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, then it is called an oscillatory sequence.

e.g: 1. $\langle a_n \rangle = \langle (-1)^n \rangle$

sequence $\langle (-1)^n \rangle$ oscillates finitely between -1 and 1.

2. $\langle a_n \rangle = \langle n(-1)^n \rangle$

sequence $\langle a_n \rangle$ oscillates infinitely between $-\infty$ and ∞.

4. Every convergent sequence has a unique limit point, i.e, a sequence cannot converge to more than one limit.

8

5. Every convergent sequence is bounded.

Monotonic Sequence:

1. A sequence $\langle a_n \rangle$ is said to be monotonically increasing if

$$a_{n+1} \geq a_n \quad \forall\ n \in \mathbb{N}$$

or $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \dots \leq a_n \leq a_{n+1} \leq \dots$

2. A sequence $\langle a_n \rangle$ is said to be monotonically decreasing if

$$a_{n+1} \leq a_n \quad \forall\ n \in \mathbb{N}$$

or $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$

3. A sequence $\langle a_n \rangle$ is said to be monotonic if it is either monotonically increasing or decreasing.

(9)

Exc: 1. $\langle a_n \rangle = \langle \frac{1}{n} \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\Rightarrow$ $\langle a_n \rangle$ converges to 0.

2. $\langle a_n \rangle = \langle \frac{1}{n^2} \rangle \to 0$

3. $\langle a_n \rangle = \langle \frac{1}{3^n} \rangle \to 0$

4. $\langle a_n \rangle = \langle 3 \rangle$: constant sequence

Range of $\langle a_n \rangle = \{3\}$ which is bounded

$\Rightarrow$ $\langle a_n \rangle$ is bounded.

5. $\langle a_n \rangle = \langle \frac{1}{2^n} \rangle$

$$0 < a_n \leq \frac{1}{2}$$

$\Rightarrow$ $\langle a_n \rangle$ is bounded.

10

6. $\langle a_n \rangle = \langle n^3 \rangle$

$a_n \geq 1$

$\langle a_n \rangle$ is Bounded below.

7. $\langle a_n \rangle = \langle -n^2 \rangle$

$a_n < 0 \quad \forall \; n \in \mathbb{N}$

$\langle a_n \rangle$ is bounded above but not bounded below.

8. $a_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$

$a_{n+1} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$

$a_{n+1} - a_n = \frac{1}{2^{n+1}} > 0 \quad \forall \; n$

$\Rightarrow a_{n+1} > a_n \quad \forall \; n$

$\Rightarrow \langle a_n \rangle$ is monotonically increasing.

and hence monotonic.

11

9. $$a_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$

$$a_{n+1} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} + \frac{1}{2n+1}$$

$$a_{n+1} - a_n = \frac{1}{2n} + \frac{1}{2n+1} - \frac{1}{n}$$

$$= \frac{-1}{2n(2n+1)} < 0$$

$\Rightarrow \quad a_{n+1} < a_n \quad \forall\, n$

$\Rightarrow \quad \langle a_n \rangle$ is monotonically decreasing and hence monotonic.

Infinite Series:

Definition: The expression of the form

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$$

is called an infinite series.

Definition: $s_n = a_1 + a_2 + \dots + a_n$ is called the partial sum of the series $\sum_{n=1}^{\infty} a_n$.

Convergence: The convergence or divergence of the $\sum_{n=1}^{\infty} a_n$ depends on the convergence or divergence of the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums.

Let $\sum_{n=1}^{\infty} a_n$ be a series with

partial sum $s_n = a_1 + a_2 + \cdots + a_n$

1. If $\langle s_n \rangle \to l$, then

$$\sum_{n=1}^{\infty} a_n \to l \quad \text{ie}$$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} a_n = l$$

2. If $\langle s_n \rangle \to \infty$, then

$$\sum_{n=1}^{\infty} a_n \to \infty$$

Result: Necessary condition for the convergence of the $\sum_{n=1}^{\infty} a_n$ is

$$\lim_{n\to\infty} a_n = 0$$

Exc: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$

$$a_n = \frac{1}{n(n+1)}$$

$$a_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2}$$

$$a_2 = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$$

$$\vdots$$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$s_n = a_1 + a_2 + \cdots + a_n$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = 1$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Exc.: $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \cdots \sqrt{\frac{n}{2(n+1)}} + \cdots$

$$a_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$

$$\lim_{n\to\infty} a_n = \frac{1}{\sqrt{2}} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ is not convergent.}$$

Result:

1. $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

2. $\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ may or may not be convergent.

3. $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ is not convergent.

KP.

Discuss the convergence of the Geometric series

$$1 + a + a^2 + \dots = \sum_{n=1}^{\infty} a^n$$

<u>Solution:</u> The partial sum of the given series is

$$S_n = 1 + a + \dots + a^{n-1}$$

$$= \frac{a^n - 1}{a - 1} \; ; \quad a \neq 1$$

<u>Case 1.</u> If $|a| < 1$, then

$$S_n = \frac{1 - a^n}{1 - a}$$

$$\lim_{n \to \infty} S_n = \frac{1}{1-a}$$

ie if $|a| < 1$, $\lim_{n \to \infty} \sum_{n=1}^{\infty} a_n = \frac{1}{1-a}$.

Case 2: If $a = 1$

$$S_n = 1 + 1 + \dots + 1 \text{ (n times)} = n$$

$$\lim_{n \to \infty} S_n = \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Case 3: If $a > 1$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a^n - 1}{a - 1} \to \infty$$

$\sum_{n=1}^{\infty} a_n$ is divergent.

Case 4: $a = -1$, then

$$s_{2n} = 1 - 1 + 1 - 1 + \cdots = 0$$

$$s_{2n+1} = -1$$

$\sum_{n=1}^{\infty} a_n$ oscillates finitely

Case 5: If $a < -1$

$$s_{2n} = \frac{a^{2n} - 1}{a - 1} \to \infty \text{ as } n \to \infty$$

$$s_{2n+1} = \frac{a^{2n+1} - 1}{a - 1} \to -\infty \text{ as } n \to \infty$$

$\sum_{n=1}^{\infty} a_n$ oscillates infinitely.

Comparison Test:

If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of positive terms, then

1) If $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and nonzero)

then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ both converge or diverge together.

2) If $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and

$\sum_{n=1}^{\infty} v_n$ converges, then $\sum_{n=1}^{\infty} u_n$ Converge

3. If $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ and

$\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} u_n$ also diverges.

Another Form of Comparison Test

1. If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of positive terms such that $u_n < v_n$ and $\sum_{n=1}^{\infty} v_n$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is also convergent.

2. If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are two series of positive terms such that $u_n > v_n$ and $\sum_{n=1}^{\infty} v_n$ is divergent then $\sum_{n=1}^{\infty} u_n$ is also divergent.

3. The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Exc: 1. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

Solution: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$$< \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$= \sum \frac{1}{n^2}$$

As $\sum \frac{1}{n^2}$ is convergent

$\therefore \sum \frac{1}{n(n+1)}$ is convergent.

2. $\sum a_n = \sum \frac{1}{n(n+3}$

$$a_n = \frac{1}{n(n+3}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n(n+3)}$$

$$= 1 \text{ (finite)}$$

As $\sum_{n=1}^{\infty} b_n$ is convergent

$\therefore \sum_{n=1}^{\infty} a_n$ is convergent

3. $$\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$$

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2n-1)n^2}{n(n+1)(n+2)} = 2$$

$\sum_{n=1}^{\infty} u_n$ is convergent as $\sum_{n=1}^{\infty} v_n$ is convergent

4. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$a_n = \frac{n}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

$\sum a_n$ is not convergent.

5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$v_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{2}$$

$\sum v_n$ is divergent

$\Rightarrow$ $\sum u_n$ is divergent.

6. $$\sum_{n=1}^{\infty} \frac{n+1}{n^p}$$

$$u_n = \frac{n+1}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$$

$$v_n = \frac{1}{n^{p-1}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

$\sum v_n$ is Convergent if $p-1>1$ ie $p>2$.

$\therefore$ $\sum u_n$ is Convergent if $p>2$.

D' Alembert's Ratio Test:

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms, then

1. $\sum_{n=1}^{\infty} u_n$ is convergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$

2. $\sum_{n=1}^{\infty} u_n$ is divergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$

3. If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$, then test fails.

Ex: 1. $1 + \frac{1^2 . 2^2}{1.3.5} + \frac{1^2 . 2^2 . 3^2}{1.3.5.7.9} + \cdots$

$$u_n = \frac{1^2 . 2^2 . 3^2 \ldots n^2}{1.3.5 \ldots (4n-5)(4n-3)}$$

$$u_{n+1} = \frac{1^2 . 2^2 . 3^2 \ldots n^2 . (n+1)^2}{1.3.5 \ldots (4n-5)(4n-3)(4n-1)(4n+1)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(4n-1)(4n+1)}{n^2} = 16 > 1$$

$\Rightarrow \sum u_n$ is convergent by D' Alembert's Ratio Test.

2. $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\dots$

Neglecting the first term

$$u_n=\frac{3\cdot 6\cdot 9\dots 3n}{7\cdot 10\cdot 13\dots(3n+4)}x^n$$

$$u_{n+1}=\frac{3\cdot 6\cdot 9\dots 3n\,(3n+3)}{7\cdot 10\cdot 13\dots(3n+4)(3n+7)}x^{n+1}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{3n+7}{3n+3}\cdot\frac{1}{x}=\frac{1}{x}$$

Σu_n converges if $\frac{1}{x}>1$ ie $x<1$

Σu_n diverges if $\frac{1}{x}<1$ i.e $x>1$

D'Alembert's ratio test fails if $x=1$.

3. $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

$\sum_{n=1}^{\infty} u_n$ is convergent if $x^2 < 1$

divergent if $x^2 > 1$

by D'Alembert's Ratio Test

Test fails if $x^2 = 1$.

If $x^2 = 1$; $u_n = \frac{1}{(n+1)\sqrt{n}}$

Take $v_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ & $\sum v_n$ is convergent

$\therefore \sum u_n$ is convergent by Comparison Test.

$$\boxed{\begin{array}{l}\underline{\underline{\text{Ans}}} \\ \text{cgt if } x^2 \le 1 \\ \text{div if } x^2 > 1\end{array}}$$

4. $$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$u_n = \frac{x^n}{n!} \quad ; \quad u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n\,(n+1)!}{n!\;x^{n+1}} \to \infty > 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent by D'Alembert's Ratio Test

5. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$u_n = \frac{n!}{n^n} \quad ; \quad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!\;(n+1)^{n+1}}{n^n\;(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent by D'Alembert's Ratio Test

6. $\sum_{n=1}^{\infty} \frac{5^n}{2^n+5}$

$$u_n = \frac{5^n}{2^n+5} \; ; \; u_{n+1} = \frac{5^{n+1}}{2^{n+1}+5}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{5^n}{2^n+5} \times \frac{2^{n+1}+5}{5^{n+1}}$$

$$= \frac{2}{5} < 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is divergent by D'Alembert's Ratio Test.

7. $\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$

$$u_n = \frac{3^n \cdot n!}{n^n} \; , \; u_{n+1} = \frac{3^{n+1} \cdot (n+1)!}{(n+1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3^n \cdot n!}{n^n} \times \frac{(n+1)^{n+1}}{3^{n+1} \cdot (n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent by D'Alembert's Ratio Test

Cauchy's Root Test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

(a) – $\lim_{n \to \infty} (u_n)^{\frac{1}{n}} = l$, then

1. $\sum_{n=1}^{\infty} u_n$ is convergent if $l < 1$
2. $\sum_{n=1}^{\infty} u_n$ is divergent if $l > 1$
3. Test fails if $l = 1$.

(b) $\lim_{n \to \infty} (u_n)^{\frac{1}{n}} = \infty$, then

$\sum_{n=1}^{\infty} u_n$ is divergent.

Exc: 1. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$(u_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^{n} = \left(1+\frac{1}{n}\right)^{-n}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = e^{-1} < 1$$

$\Rightarrow$ $\sum u_n$ is convergent.

2. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

$$u_n = \frac{x^n}{n^n}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x}{n} = 0 < 1$$

$\Rightarrow$ $\sum u_n$ is Convergent.

3. $\sum_{n=1}^{\infty} 5^{-n-(-1)^n}$

$$u_n = 5^{-n-(-1)^n}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} 5^{-1-\frac{(-1)^n}{n}} = \lim_{n\to\infty} \frac{5^{\frac{(-1)^{n+1}}{n}}}{5}$$

$$= \frac{1}{5} < 1$$

$\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent.

4. $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} \cdot x^n$

$$u_n = \frac{(n+1)^n}{n^{n+1}} \cdot x^n$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{x}{\sqrt[n]{n}}$$

$$= x \quad \left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$$

① $\sum_{n=1}^{\infty} u_n$ is convergent if

$x < 1$ by Cauchy root test.

② $\sum_{n=1}^{\infty} u_n$ is divergent if $x > 1$

③ Test fails if $x = 1$

When $x = 1$

$$u_n = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

$$\cancel{\lim_{n\to\infty} (u_n)^{\frac{1}{n}}} \qquad v_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$\sum_{n=1}^{\infty} u_n$ is divergent as $\sum_{n=1}^{\infty} v_n$ is divergent.

5. $$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$$

$$u_n = \left(1+\frac{1}{n}\right)^{-n^2}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{e} < 1$$

$\Rightarrow$ $\sum u_n$ is convergent by Cauchy Root test.

Cauchy Root Test is more general than D'Alembert's Ratio Test :

1. $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ exists

$\Rightarrow \lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}}$ exists.

$\therefore \lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}}$

2. $\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}}$ exists need not imply $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$

ie $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ may or may not exists.

Exc:

$$u_n = \begin{cases} 2^{-n} & n \text{ is odd} \\ 2^{-n+2} & n \text{ is even} \end{cases}$$

Case 1: n is odd

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(2^{-n}\right)^{\frac{1}{n}} = \frac{1}{2} < 1$$

Case 2: n is even

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(2^{-n+2}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2^{2/n}}{2} = \frac{1}{2} < 1$$

By cauchy Root test, $\sum u_n$ is Convergent.

Case 1: n is odd

$$u_n = 2^{-n}$$

$$u_{n+1} = 2^{-(n+1)+2} = 2^{-n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

Case 2: n is even

$$u_n = 2^{-n+2}$$

$$u_{n+1} = 2^{-(n+1)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} 2^3 = 8 > 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} \text{ does not exist.}$$

Raabe's Test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l,$$

then

1. $\sum_{n=1}^{\infty} u_n$ is convergent if $l > 1$
2. $\sum_{n=1}^{\infty} u_n$ is divergent if $l < 1$
3. Test fails if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$

Note: Raabe's Test is stronger than D'Alembert's Ratio test may succeed whose Ratio test fails.

Example: $\sum \frac{1}{n^2}$

$$u_n = \frac{1}{n^2}$$

$$u_{n+1} = \frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2$$

$$= 1$$

Ratio Test fails

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(n+1)^2}{n^2} - 1\right)$$

$$= \lim_{n\to\infty} \frac{1+2n}{n}$$

$$= 2 > 1$$

$\sum_{n=1}^{\infty} u_n$ is convergent by Raabe's Test.

Note

1. $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \infty$$

$\Rightarrow$ $\sum u_n$ is convergent.

2. $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = -\infty$$

$\Rightarrow$ $\sum u_n$ is divergent.

3. Raabe's test is used when D'Alembert's Ratio test fails and ratio "$\frac{u_n}{u_{n+1}}$" does not involve the number "e".

4. When the Ratio "$\frac{u_n}{u_{n+1}}$" involves "e" we will apply Logarithmic Test after the Ratio Test not Raabe's Test.

Logarithmic Test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l \,,$$

then

(1) $\sum_{n=1}^{\infty} u_n$ is convergent if $l > 1$

(2) $\sum_{n=1}^{\infty} u_n$ is divergent if $l < 1$

(3) The test fails if $l = 1$.

Exc: 1.

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots$$

Solution: Neglecting the first term,

we have

$$u_n = \frac{n!\; x^n}{(n+1)^n}$$

$$u_{n+1} = \frac{(n+1)!\; x^{n+1}}{(n+2)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!\; x^n}{(n+1)^n} \cdot \frac{(n+2)^{n+1}}{(n+1)!\; x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{1}{x}$$

$$= \frac{e}{x}$$

By Ratio Test
1. Σu_n Converges if $x < e$
2. Σu_n diverges if $x > e$
3. Test fails if $x = e$

When $x = e$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1 \quad , \quad \frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{1}{e}$$

$$\log \frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\frac{2}{n} - \frac{1}{2}\left(\frac{4}{n^2}\right) + \frac{1}{3}\left(\frac{8}{n^3}\right) - \cdots\right] - (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \cdots\right] - 1$$

$$= \frac{-1}{2n} - \frac{3}{2n^2} + \cdots$$

$$n \log \frac{u_n}{u_{n+1}} = \frac{-1}{2} - \frac{3}{2n} + \cdots$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \frac{-1}{2} < 1$$

By Logarithmic Test the $\sum_{n=1}^{\infty} u_n$ is divergent if $x = e$.

Gauss Test: If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

1. $\sum_{n=1}^{\infty} u_n$ is convergent if $\lambda > 1$

2. $\sum_{n=1}^{\infty} u_n$ is divergent if $\lambda \leq 1$.

Note: This test is applied after the failure of Ratio test and when it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $\frac{1}{n}$ by Binomial theorem or by any other method.

Exc: $$\sum_{n=1}^{\infty} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}$$

$$u_n = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}$$

$$u_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2 (2n+2)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$

$\Rightarrow$ Ratio Test fails

Now, $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\left(\frac{2n+2}{2n+1}\right)^2 - 1\right]$$

$$= \frac{4n^2+3n}{(2n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$$

$\Rightarrow$ Raabe's Test fails.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$= \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{1}{n}+\frac{3}{4n^2}-\cdots\right)$$

$$= 1+\frac{1}{n}+\frac{(-1/4)}{n^2}+\cdots$$

$$= 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

$\lambda = 1$

$\sum_{n=1}^{\infty} u_n$ is divergent by Gauss Test.

Exc: $1 + a + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \cdots$

$a > 0$

Neglecting the first term, we have

$$u_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{1\cdot 2\cdot 3\cdots n}$$

$$u_{n+1} = \frac{a\cdot(a+1)\cdots\quad a+n}{1\cdot 2\cdot 3\cdots\quad (n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1 \quad (\text{Ratio Test fails})$$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{a}{n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{a}{n}+\frac{a^2}{n^2}-\cdots\right)$$

$$= 1 + \frac{1-a}{n} + O\left(\frac{1}{n^2}\right)$$

$\lambda = 1-a$

Σu_n is convergent if $1-a > 1 \Rightarrow a < 0$

Σu_n is divergent if $1-a \leq 1 \Rightarrow a \geq 0$

Cauchy's Integral Test

If $f(x)$ is non-negative monotonically decreasing integrable function such that

$$f(n) = a_n$$

integral

for all positive ~~integrable~~ values of n, then the series

$\sum_{n=1}^{\infty} a_n$ and improper integral

$\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Exc: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$u_n = \frac{1}{n^2+1} = f(n)$$

$$\therefore \quad f(x) = \frac{1}{x^2+1}$$

for $x \geq 1$, $f(x) \geq 0$ monotonically decreasing function of x.

$\therefore$ Cauchy Integral test is applicable

$$I_n = \int_1^n f(x)\,dx$$

$$= \int_1^n \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1}x\right]_1^n = \tan^{-1}n - \frac{\pi}{4}$$

$$\lim_{n\to\infty} I_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$\Rightarrow \int_1^x f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} u_n$ is convergent by Cauchy Integral Test.

Show that $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Proof: Here

$$u_n = \frac{1}{n^p} = f(n)$$

$$\therefore \quad f(x) = \frac{1}{x^p}$$

for $x \geq 1$ and $p > 0$, $f(x)$ is positive and decreasing.

$\therefore$ Cauchy Integral test is applicable.

Case 1: $p \neq 1$

$$I_n = \int_1^n f(x)\,dx = \int_1^n \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^n$$

Subcase 1: $p > 1$, $p - 1 > 0$

$$\therefore \quad I_n = \frac{-1}{p-1}\left[\frac{1}{x^{p-1}}\right]_1^n$$

$$= \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right]$$

$$\lim_{n \to \infty} I_n = \frac{1}{p-1} = \text{finite}$$

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \frac{1}{p-1} = \text{finite}$$

$\Rightarrow \int_1^\infty f(x)\,dx$ converges

$\Rightarrow \sum_{n=1}^\infty u_n$ converges

<u>Subcase (ii)</u> When $0 < p < 1 \Rightarrow 1-p > 0$

$$I_n = \frac{1}{1-p}\left[n^{1-p} - 1\right]$$

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n$$

$$= \lim_{n\to\infty} \frac{1}{1-p}\left[n^{1-p} - 1\right]$$

$$= \infty$$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges

$\Rightarrow \sum_{n=1}^\infty u_n$ diverges.

Case 2: $p = 1$

$$f(x) = \frac{1}{x}$$

$$I_n = \int_1^n f(x)\,dx$$

$$= \int_1^n \frac{1}{x}\,dx$$

$$= \left[\log x\right]_1^n$$

$$= \log n - \log 1 = \log n$$

$$\therefore \int_1^\infty f(x)\,dx = \lim_{n\to\infty} I_n = \infty$$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^\infty u_n \text{ diverges}$$

Hence $\sum_{n=1}^\infty u_n$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Alternating Series:

A series with terms alternately positive and negative is called alternating series.

Thus the series

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \dots$$

can be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n \; ; \quad u_n > 0$$

Leibnitz Test on Alternating Series:

The alternating series $\sum_{n=1}^{\infty} (-1)^n u_n$ converges if

1. $u_n \geq u_{n+1} \quad \forall n$

2. $\lim_{n \to \infty} u_n = 0$

Absolute Convergence: A series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

Conditionally convergent: If $\sum_{n=1}^{\infty} u_n$ converges but not absolutely then $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Result: Every absolutely convergent series is convergent.

Exc

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$; $u_n = (-1)^{n+1} v_n$

Here $v_n = \frac{1}{n} > 0 \;\; \forall\, n$

$$v_{n+1} = \frac{1}{n+1}$$

$$v_n > v_{n+1} \quad \forall\, n$$

Also $\lim_{n\to\infty} v_n = 0$

By Leibnitz test, $\sum (-1)^{n+1} v_n$ is convergent.

Now $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} |u_n|$ is not convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

$$v_n = \frac{1}{2n-1}$$

$$v_{n+1} = \frac{1}{2n}$$

Also $v_n > 0 \quad \forall n$

and $v_n > v_{n+1} \quad \forall n$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} v_n$ is convergent by Leibnitz Test.

$$u_n = (-1)^{n-1} v_n$$

$$|u_n| = \frac{1}{2n-1}$$

$\sum_{n=1}^{\infty} |u_n|$ is divergent

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2n-1}$ is conditionally Convergent.

3. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n}{2n-1} = \sum_{n=1}^{\infty} u_n$$

$$u_n = (-1)^{n-1} \cdot v_n$$

$$v_n = \frac{n}{2n-1} > 0$$

$$v_n > v_{n+1}$$

but $$\lim_{n\to\infty} v_n = \frac{1}{2} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} u_n \text{ is not convergent.}$$

4. $$\sum_{n=1}^{\infty} (-1)^n \frac{(n+5)}{n(n+1)}$$

5. $$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$$

$$u_n = (-1)^n \frac{2^n}{n!} \quad ; \quad u_{n+1} = (-1)^{n+1} \frac{2^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim_{n\to\infty} \frac{2^n}{n!} \frac{(n+1)!}{2^{n+1}} = \infty > 1$$

$\Rightarrow \sum |u_n|$ is convergent $\Rightarrow \sum u_n$ is Convergent

6.

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n^{100}}{2n!}$$

$$|u_n| = \frac{n^{100}}{2n!}$$

$$|u_{n+1}| = \frac{(n+1)^{100}}{(2n+2)!}$$

$$\lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \frac{(2n+2)! \cdot n^{100}}{(2n)!\,(n+1)^{100}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{\left(1+\frac{1}{n}\right)^{100}}$$

$$= \infty > 1$$

$$\Rightarrow \sum_{n=1}^{\infty} |u_n| \text{ is Convergent}$$

$$\Rightarrow \sum u_n \text{ is convergent.}$$