Example 1

Three identical coils each of [4.2 + j5.6] ohms are connected in star across a 415 V, 3-phase, 50 Hz supply. Determine (i) V_{ph}, (ii) I_{ph}, and (iii) power factor. **[May 2014]**

Solution

$$\bar{Z}_{ph} = 4.2 + j5.6 = 7 \angle 53.13° \ \Omega$$
$$V_L = 415 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a star-connected load,

(i) $$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6 \text{ V}$$

(ii) $$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{239.6}{7} = 34.23 \text{ A}$$

(iii) $$\text{pf} = \cos\phi = \cos(53.13°) = 0.6 \text{ (lagging)}$$

Example 2

Three equal impedances, each of 8 + j10 ohms, are connected in star. This is further connected to a 440 V, 50 Hz, three-phase supply. Calculate (i) phase voltage, (ii) phase angle, (iii) phase current, (iv) line current, (v) active power, and (vi) reactive power.

Solution

$$\bar{Z}_{ph} = 8 + j10 \ \Omega$$
$$V_L = 440 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a star-connected load,

(i) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$

(ii) Phase angle

$$\bar{Z}_{ph} = 8 + j10 = 12.81 \angle 51.34° \ \Omega$$
$$Z_{ph} = 12.81 \ \Omega$$
$$\phi = 51.34°$$

(iii) Phase current

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{254.03}{12.81} = 19.83 \text{ A}$$

(iv) Line current

$$I_L = I_{ph} = 19.83 \text{ A}$$

(v) Active power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 440 \times 19.83 \times \cos(51.34°) = 9.44 \text{ kW}$$

(vi) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 440 \times 19.83 \times \sin(51.34°) = 11.81 \text{ kVAR}$$

Example 3

A balanced delta-connected load of impedance (8 – j6) ohms per phase is connected to a three-phase, 230 V, 50 Hz supply. Calculate (i) power factor, (ii) line current, and (iii) reactive power.

Solution

$$\bar{Z}_{ph} = 8 - j6\ \Omega$$
$$V_L = 230 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a delta-connected load,

(i) Power factor

$$\bar{Z}_{ph} = 8 - j6 = 10 \angle -36.87°\ \Omega$$
$$Z_{ph} = 10\ \Omega$$
$$\phi = 36.87°$$
$$\text{pf} = \cos\phi = \cos(36.87°) = 0.8 \text{ (leading)}$$

(ii) Line current

$$V_{ph} = V_L = 230 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230}{10} = 23 \text{ A}$$
$$I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 23 = 39.84 \text{ A}$$

(iii) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 230 \times 39.84 \times \sin(36.87°) = 9.52 \text{ kVAR}$$

Example 4

Three coils, each having a resistance and an inductance of 8 Ω and 0.02 H respectively, are connected in star across a three-phase, 230 V, 50 Hz supply. Find the (i) power factor, (ii) line current, (iii) power, (iv) reactive volt-amperes, and (v) total volt-amperes.

Solution

$$R = 8\ \Omega$$
$$L = 0.02 \text{ H}$$
$$V_L = 230 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a star-connected load,

(i) Power factor

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$
$$\bar{Z}_{ph} = R + jX_L = 8 + j6.28 = 10.17\ \angle 38.13°\ \Omega$$
$$Z_{ph} = 10.17\ \Omega$$
$$\phi = 38.13°$$
$$\text{pf} = \cos\phi = \cos(38.13°) = 0.786 \text{ (lagging)}$$

(ii) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{230}{\sqrt{3}} = 132.79\ \text{V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{132.79}{10.17} = 13.05\ \text{A}$$
$$I_L = I_{ph} = 13.05\ \text{A}$$

(iii) Power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 230 \times 13.05 \times 0.786 = 4.088\ \text{kW}$$

(iv) Reactive volt-amperes

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 230 \times 13.05 \times \sin(38.13°) = 3.21\ \text{kVAR}$$

(v) Total volt-ampere

$$S = \sqrt{3}\ V_L I_L = \sqrt{3} \times 230 \times 13.05 = 5.198\ \text{kVA}$$

Example 5

Three similar coils each having a resistance of 10 Ω and inductance of 0.04 H are connected in star across a 3 phase, 50 Hz, 200 V supply. Calculate the line current, total power absorbed, reactive volt amperes and total volt amperes. **[May 2015]**

Solution

$$R = 10\ \Omega$$
$$L = 0.04\ \text{H}$$
$$V_L = 200\ \text{V}$$
$$f = 50\ \text{Hz}$$
$$X_L = 2\pi fL = 2\pi \times 50 \times 0.04 = 12.57\ \Omega$$
$$Z_{ph} = R + j\,X_L = 10 + j\,12.57 = 16.06\ \angle\ 51.5°\ \Omega$$

(i) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{200}{\sqrt{3}} = 115\cdot 47\ \text{V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{115 \cdot 47}{16.06} = 7.19 \text{ A}$$

$$I_L = I_{ph} = 7.19 \text{ A}$$

(ii) Total power absorbed

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 200 \times 7.19 \times \cos(51.5°) = 1550.5 \text{ W}$$

(iii) Reactive volt-ampere

$$Q = \sqrt{3}\, V_L I_L \sin\phi = \sqrt{3} \times 200 \times 7.19 \times \sin(51.5°) = 1949.23 \text{ VAR}$$

(iv) Total volt ampere

$$S = \sqrt{3}\, V_L I_L = \sqrt{3} \times 200 \times 7.19 = 2490.68 \text{ VA}$$

Example 6

Three coils, each having a resistance of 8 Ω and an inductance of 0.02 H, are connected in delta to a three-phase, 400 V, 50 Hz supply. Calculate the (i) line current, and (ii) power absorbed.

Solution

$$R = 8\ \Omega$$
$$L = 0.02 \text{ H}$$
$$V_L = 400 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a delta-connected load,

(i) Line current

$$V_L = V_{ph} = 400 \text{ V}$$
$$X_L = 2\pi fL = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$
$$\bar{Z}_{ph} = R + jX_L = 8 + j6.28 = 10.17 \angle 38.13°\ \Omega$$
$$Z_{ph} = 10.17\ \Omega$$
$$\phi = 38.13°$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{10.17} = 651639.33 \text{ A}$$
$$I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 39.33 = 68.12 \text{ A}$$

(ii) Power absorbed

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 400 \times 68.12 \times \cos(38.13°) = 37.12 \text{ kW}$$

Example 7

The three equal impedances of each of 10 ∠60° Ω, are connected in star across a three-phase, 400 V, 50 Hz supply. Calculate the (i) line voltage and phase voltage, (ii) power factor and active power consumed, (iii) If the same three impedances are connected in delta to the same source of supply, what is the active power consumed?

Solution $\bar{Z}_{ph} = 10 \angle 60° \ \Omega$

$V_L = 400$ V

$f = 50$ Hz

For a star-connected load,

(i) Line voltage and phase voltage

$$V_L = 400 \text{ V}$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

(ii) Power factor and active power consumed

$$\phi = 60°$$

$$\text{pf} = \cos\phi = \cos(60°) = 0.5 \text{ (lagging)}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{10} = 23.094 \text{ A}$$

$$I_L = I_{ph} = 23.094 \text{ A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 23.094 \times 0.5 = 8 \text{ kW}$$

(iii) Active power consumed for delta-connected load

$$V_L = 400 \text{ V}$$

$$Z_{ph} = 10 \ \Omega$$

$$V_{ph} = V_L = 400 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{10} = 40 \text{ A}$$

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 40 = 69.28 \text{ A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 69.28 \times \cos(60°) = 24 \text{ kW}$$

Example 8

Three similar coils A, B, and C are available. Each coil has a 9 Ω resistance and a 12 Ω reactance. They are connected in delta to a three-phase, 440 V, 50 Hz supply. Calculate for this load, the (i) phase current, (ii) line current, (iii) power factor, (iv) total kVA, (v) active power, and (vi) reactive power. If these coils are connected in star across the same supply, calculate all the above quantities.

Solution $R = 9 \ \Omega$

$X_L = 12 \ \Omega$

$V_L = 440$ V

$f = 50$ Hz

For a delta-connected load,

(i) Phase current

$$V_L = V_{ph} = 440 \text{ V}$$

$$\bar{Z}_{ph} = 9 + j12 = 15 \angle 53.13° \ \Omega$$

$$Z_{ph} = 15 \ \Omega$$

$$\phi = 53.13°$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{440}{15} = 29.33 \text{ A}$$

(ii) Line current

$$I_L = \sqrt{3} \ I_{ph} = \sqrt{3} \times 29.33 = 50.8 \text{ A}$$

(iii) Power factor

$$\text{pf} = \cos \phi = \cos (53.13°) = 0.6 \text{ (lagging)}$$

(iv) Total kVA

$$S = \sqrt{3} \ V_L I_L = \sqrt{3} \times 440 \times 50.8 = 38.71 \text{ kVA}$$

(v) Active power

$$P = \sqrt{3} \ V_L I_L \cos \phi = \sqrt{3} \times 440 \times 50.8 \times 0.6 = 23.23 \text{ kW}$$

(vi) Reactive power

$$Q = \sqrt{3} \ V_L I_L \sin \phi = \sqrt{3} \times 440 \times 50.8 \times \sin (53.13°) = 30.97 \text{ kVAR}$$

If these coils are connected in star across the same supply,

(i) Phase current

$$V_L = 440 \text{ V}$$

$$Z_{ph} = 15 \ \Omega$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{254.03}{15} = 16.94 \text{ A}$$

(ii) Line current

$$I_L = I_{ph} = 16.94 \text{ A}$$

(iii) Power factor

$$\text{pf} = 0.6 \text{ (lagging)}$$

(iv) Total kVA

$$S = \sqrt{3} \ V_L I_L = \sqrt{3} \times 440 \times 16.94 = 12.91 \text{ kVA}$$

(v) Active power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 440 \times 16.94 \times 0.6 = 7.74 \text{ kW}$$

(vi) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 440 \times 16.94 \times \sin(53.13°) = 12.33 \text{ kVAR}$$

Example 9

A balanced 3-phase load consists of 3 coils, each of resistance 4 Ω and inductance 0.02 H. It is connected to a 440 V, 50 Hz, 3 ϕ supply. Find the total power consumed when the load is connected in star and the total reactive power when the load is connected in delta. **[Dec 2014]**

Solution $R = 4\ \Omega$

$L = 0.02$ H

$V_L = 440$ V

$f = 50$ Hz

For a star-connected load,

(i) Total power consumed

$$X_L = 2\pi f L = 2\pi \times 50 \times 0.02 = 6.28\ \Omega$$

$$\bar{Z}_{ph} = R + j X_L = 4 + j\,6.28 = 7.45 \angle 57.51°\ \Omega$$

$$Z_{ph} = 7.45\ \Omega$$

$$\phi = 57.51°$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{254.03}{7.45} = 34.1 \text{ A}$$

$$I_L = I_{ph} = 34.1 \text{ A}$$

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 440 \times 34.1 \times \cos(57.51°) = 13.96 \text{ kW}$$

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 440 \times 34.1 \times \sin(57.51°) = 21.92 \text{ kVAR}$$

(ii) When the load is connected in delta across same supply

$$Q_\Delta = 3Q_Y = 3 \times 21.92 \times 10^3 = 65.76 \text{ kVAR}$$

Example 10

A 415 V, 50 Hz, three-phase voltage is applied to three star-connected identical impedances. Each impedance consists of a resistance of 15 Ω, a capacitance of 177 µF and an inductance of 0.1 henry in series. Find the (i) power factor, (ii) phase current, (iii) line current, (iv) active power, (v) reactive power, and (vi) total VA. Draw a neat phasor diagram. If the same impedances are connected in delta, find the (i) line current, and (ii) power consumed. **[Dec 2015]**

Solution

$$V_L = 415 \text{ V}$$
$$f = 50 \text{ Hz}$$
$$R = 15\ \Omega$$
$$C = 177\ \mu\text{F}$$
$$L = 0.1 \text{ H}$$

For a star-connected load,

(i) Power factor

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$
$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 177 \times 10^{-6}} = 17.98\ \Omega$$
$$\bar{Z}_{ph} = R + jX_L - jX_C$$
$$= 15 + j31.42 - j17.98$$
$$= 15 + j13.44$$
$$= 20.14\ \angle 41.86°\ \Omega$$
$$Z_{ph} = 20.14\ \Omega$$
$$\phi = 41.86°$$
$$\text{pf} = \cos\phi = \cos(41.86°) = 0.744 \text{ (lagging)}$$

(ii) Phase current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{239.6}{20.14} = 11.9 \text{ A}$$

(iii) Line current

$$I_L = I_{ph} = 11.9 \text{ A}$$

(iv) Active power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 415 \times 11.9 \times 0.744 = 6.36 \text{ kW}$$

(v) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 415 \times 11.9 \times \sin(41.86°) = 5.71 \text{ kVAR}$$

(vi) Total VA

$$S = \sqrt{3}\ V_L I_L = \sqrt{3} \times 415 \times 11.9 = 8.55 \text{ kVA}$$

Phasor Diagram

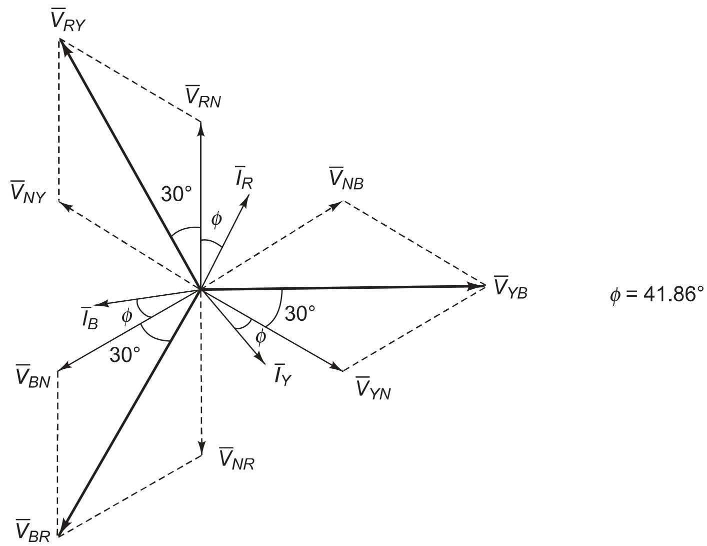


Fig. 5.21

If the same impedances are connected in delta,

(i) Line current

$$V_L = V_{ph} = 415 \text{ V}$$

$$Z_{ph} = 20.14\ \Omega$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{415}{20.14} = 20.61 \text{ A}$$

$$I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 20.61 = 35.69 \text{ A}$$

(ii) Power consumed

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 415 \times 35.69 \times 0.744 = 19.09 \text{ kW}$$

Example 11

Each phase of a delta-connected load consists of a 50 mH inductor in series with a parallel combination of a 50 Ω resistor and a 50 µF capacitor. The load is connected to a three-phase, 550 V, 800 rad/s ac supply. Find the (i) power factor, (ii) phase current, (iii) line current, (iv) power drawn, (v) reactive power, and (vi) kVA rating of the load.

Solution

$$L = 50 \text{ mH}$$

$$R = 50\ \Omega$$

$$C = 50\ \mu\text{F}$$

$$V_L = 550 \text{ V}$$
$$\omega = 800 \text{ rad/s}$$

For a delta-connected load,

(i) Power factor

$$X_L = \omega L = 800 \times 50 \times 10^{-3} = 40\ \Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{800 \times 50 \times 10^{-6}} = 25\,\Omega$$
$$\bar{Z}_{ph} = jX_L + \frac{R(-jX_C)}{R - jX_C}$$
$$= j40 + \frac{50(-j25)}{50 - j25}$$
$$= 10 + j20 = 22.36\ \angle 63.43°\ \Omega$$
$$Z_{ph} = 22.36\ \Omega$$
$$\phi = 63.43°$$
$$\text{pf} = \cos\phi = \cos(63.43°) = 0.447 \text{ (lagging)}$$

(ii) Phase current

$$V_L = V_{ph} = 550 \text{ V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{550}{22.36} = 24.6\ \text{A}$$

(iii) Line current

$$I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 24.6 = 42.61 \text{ A}$$

(iv) Power drawn

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 550 \times 42.61 \times 0.447 = 18.14 \text{ kW}$$

(v) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 550 \times 42.61 \times \sin(63.43°) = 36.3 \text{ kVAR}$$

(vi) kVA rating of the load

$$S = \sqrt{3}\ V_L I_L = \sqrt{3} \times 550 \times 42.61 = 40.59 \text{ kVA}$$

Example 12

A balanced star-connected load is supplied from a symmetrical three-phase 400 volts, 50 Hz system. The current in each phase is 30 A and lags 30° behind the phase voltage. Find the (i) phase voltage, (ii) resistance and reactance per phase, (iii) load inductance per phase, and (iv) total power consumed.

Solution $V_L = 400$ V

$$f = 50 \text{ Hz}$$
$$I_{ph} = 30 \text{ A}$$
$$\phi = 30°$$

For a star-connected load,

(i) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

(ii) Resistance and reactance per phase

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{30} = 7.7\,\Omega$$
$$Z_{ph} = Z_{ph}\angle\phi = 7.7\angle 30° = (6.67 + j\,3.85)\,\Omega$$
$$R_{ph} = 6.67\,\Omega$$
$$X_{ph} = 3.85\,\Omega$$

(iii) Load inductance per phase

$$X_{ph} = 2\pi f L_{ph}$$
$$3.85 = 2\pi \times 50 \times L_{ph}$$
$$L_{ph} = 0.01225 \text{ H}$$

(iv) Total power consumed

$$P = 3V_{ph} I_{ph} \cos\phi = 3 \times 230.94 \times 30 \times \cos(30°) = 18 \text{ kW}$$

Example 13

A symmetrical three-phase 400 V system supplies a basic load of 0.8 lagging power factor and is connected in star. If the line current is 34.64 A, find the (i) impedance, (ii) resistance and reactance per phase, (iii) total power, and (iv) total reactive voltamperes.

Solution

$$V_L = 400 \text{ V}$$
$$\text{pf} = 0.8 \text{ (lagging)}$$
$$I_L = 34.64 \text{ A}$$

For a star-connected load,

(i) Impedance

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$
$$I_{ph} = I_L = 34.64 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{34.64} = 6.67\,\Omega$$

(ii) Resistance and reactance per phase

$$\text{pf} = \cos\phi = 0.8$$

$$\phi = \cos^{-1}(0.8) = 36.87°$$

$$Z_{ph} = Z_{ph}\angle\phi = 6.67\ \angle 36.87° = (5.33 + j\,4)\ \Omega$$

$$R_{ph} = 5.33\ \Omega$$

$$X_{ph} = 4\ \Omega$$

(iii) Total power

$$P = \sqrt{3}\,V_L I_L \cos\phi = \sqrt{3} \times 400 \times 34.64 \times 0.8 = 19.19\ \text{kW}$$

(iv) Total reactive volt-amperes

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 400 \times 34.64 \times \sin(36.87°) = 14.4\ \text{kVAR}$$

Example 14

A balanced star-connected load is supplied by a 415 V, 50 Hz three-phase system. Current in each phase is 20 A and lags 30° behind its phase voltage. Find the (i) phase voltage, (ii) power, and (iii) circuit parameters. Also, find power consumed when the same load is connected in delta across the same supply.

Solution

$$V_L = 415\ \text{V}$$

$$f = 50\ \text{Hz}$$

$$I_{ph} = 20\ \text{A}$$

$$\phi = 30°$$

For a star-connected load,

(i) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{415}{\sqrt{3}} = 239.6\ \text{V}$$

(ii) Power

$$I_L = I_{ph} = 20\ \text{A}$$

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 415 \times 20 \times \cos(30°) = 12.45\ \text{kW}$$

(iii) Circuit parameters

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{239.6}{20} = 11.98\ \Omega$$

$$\bar{Z}_{ph} = Z_{ph}\angle\phi = 11.98\ \angle 30° = (10.37 + j6)\ \Omega$$

$$R_{ph} = 10.37\ \Omega$$

$$X_{ph} = 6\ \Omega$$

$$X_{ph} = 2\pi f L_{ph}$$

$$6 = 2\pi \times 50 \times L_{ph}$$

$$L_{ph} = 19.1 \text{ mH}$$

(iv) Power consumed by same delta load across the same supply

$$P_\Delta = 3P_Y = 3 \times 12.45 \times 10^3 = 37.35 \text{ kW}$$

Example 15

Three identical coils connected in delta to a 440 V, three-phase supply take a total power of 50 kW and a line current of 90 A. Find the (i) phase current, (ii) power factor, and (iii) apparent power taken by the coils.

Solution $V_L = 440 \text{ V}$

$$P = 50 \text{ kW}$$

$$I_L = 90 \text{ A}$$

For a delta-connected load,

(i) Phase current

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{90}{\sqrt{3}} = 51.96 \text{ A}$$

(ii) Power factor

$$P = \sqrt{3}\, V_L\, I_L \cos \phi$$

$$50 \times 10^3 = \sqrt{3} \times 440 \times 90 \times \cos \phi$$

$$\text{pf} = \cos \phi = 0.73 \text{ (lagging)}$$

(iii) Apparent power

$$S = \sqrt{3}\, V_L\, I_L = \sqrt{3} \times 440 \times 90 = 68.59 \text{ kVA}$$

Example 16

Three similar choke coils are connected in star to a three-phase supply. If the line current is 15 A, the total power consumed is 11 kW and the volt-ampere input is 15 kVA, find the line and phase voltages, the VAR input and the reactance and resistance of each coil. If these coils are now connected in delta to the same supply, calculate phase and line currents, active and reactive power.

Solution $I_L = 15 \text{ A}$

$$P = 11 \text{ kW}$$

$$S = 15 \text{ kVA}$$

For a star-connected load,

(i) Line voltage

$$S = \sqrt{3}\, V_L\, I_L$$

$$15 \times 10^3 = \sqrt{3} \times V_L \times 15$$
$$V_L = 577.35 \text{ V}$$

(ii) Phase voltage

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{577.35}{\sqrt{3}} = 333.33 \text{ V}$$

(iii) VAR input

$$\cos\phi = \frac{P}{S} = \frac{11 \times 10^3}{15 \times 10^3} = 0.733$$
$$\phi = 42.86°$$
$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 577.35 \times 15 \times \sin(42.86°) = 10.2 \text{ kVAR}$$

(iv) Reactance and resistance of coil

$$I_{ph} = I_L = 15 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{333.33}{15} = 22.22\ \Omega$$
$$R = Z_{ph} \cos\phi = 22.22 \times 0.733 = 16.29\ \Omega$$
$$X_L = Z_{ph} \sin\phi = 22.22 \times \sin(42.86°) = 15.11\ \Omega$$

If these coils are now connected in delta,

(i) Phase current

$$V_{ph} = V_L = 577.35 \text{ V}$$
$$Z_{ph} = 22.22\ \Omega$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{577.35}{22.22} = 25.98 \text{ A}$$

(ii) Line current

$$I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 25.98 = 45 \text{ A}$$

(iii) Active power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 577.35 \times 45 \times 0.733 = 32.98 \text{ kW}$$

(iv) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 577.35 \times 45 \times \sin(42.86°) = 30.61 \text{ kVAR}$$

Example 17

Three similar coils, connected in star, take a total power of 1.5 kW at p.f. of 0.2 lagging from a three-phase, 440 V, 50 Hz supply. Calculate the resistance and inductance of each coil. **[Dec 2012]**

Solution

$$P = 1.5 \text{ kW}$$
$$\text{pf} = 0.2 \text{ (lagging)}$$
$$V_L = 440 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a star-connected load.

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.03 \text{ V}$$
$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$1.5 \times 10^3 = \sqrt{3} \times 440 \times I_L \times 0.2$$
$$I_L = 9.84 \text{ A}$$
$$I_{ph} = I_L = 9.84 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{254.03}{9.84} = 25.82\ \Omega$$
$$\phi = \cos^{-1}(0.2) = 78.46°$$
$$\bar{Z}_{ph} = Z_{ph}\angle\phi = 25.82\angle 78.46° = (5.17 + j25.3)\ \Omega$$
$$R_{ph} = 5.17\ \Omega$$
$$X_{L_{ph}} = 25.3\ \Omega$$
$$X_{L_{ph}} = 2\pi f L_{ph}$$
$$25.3 = 2\pi \times 50 \times L_{ph}$$
$$L_{ph} = 0.08 \text{ H}$$

Example 18

A three-phase, star-connected source feeds 1500 kW at 0.85 power factor lag to a balanced mesh-connected load. Calculate the current, its active and reactive components in each phase of the source and the load. The line voltage is 2.2 kV.

Solution

$$P = 1500 \text{ kW}$$
$$\text{pf} = 0.85 \text{ (lagging)}$$
$$V_L = 2.2 \text{ kV}$$

For a mesh or delta-connected load,

(i) Line current

$$P = \sqrt{3}\, V_L I_L \cos\phi$$
$$1500 \times 10^3 = \sqrt{3} \times 2.2 \times 10^3 \times I_L \times 0.85$$
$$I_L = 463.12 \text{ A}$$

(ii) Active component of current in each phase of the load

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{463.12}{\sqrt{3}} = 267.38 \text{ A}$$

$$I_{ph} \cos \phi = 267.38 \times 0.85 = 227.27 \text{ A}$$

(iii) Reactive component of current in each phase of the load

$$I_{ph} \sin \phi = 267.38 \times \sin (\cos^{-1} 0.85)$$
$$= 267.38 \times 0.526 = 140.85 \text{ A}$$

For a star-connected source, the phase current in the source will be the same as the line current drawn by the load.

(iv) Active component of this current in each phase of the source

$$I_L \cos \phi = 463.12 \times 0.85 = 393.65 \text{ A}$$

(v) Reactive component of this current in each phase of the source

$$I_L \sin \phi = 463.12 \times 0.526 = 243.6 \text{ A}$$

Example 19

A three-phase, 208-volt generator supplies a total of 1800 W at a line current of 10 A when three identical impedances are arranged in a Wye connection across the line terminals of the generator. Compute the resistive and reactive components of each phase impedance.

Solution

$$V_L = 208 \text{ V}$$
$$P = 1800 \text{ W}$$
$$I_L = 10 \text{ A}$$

For a Wye-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{208}{\sqrt{3}} = 120.09 \text{ V}$$

$$I_{ph} = I_L = 10 \text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{120.09}{10} = 12\ \Omega$$

$$P = \sqrt{3}\ V_L I_L \cos \phi$$

$$1800 = \sqrt{3} \times 208 \times 10 \times \cos \phi$$

$$\cos \phi = 0.5$$

$$\phi = 60°$$

$$R_{ph} = Z_{ph} \cos \phi = 12 \times 0.5 = 6\ \Omega$$

$$X_{ph} = Z_{ph} \sin \phi = 12 \times \sin (60°) = 10.39\ \Omega$$

Example 20

A balanced, three-phase, star-connected load of 100 kW takes a leading current of 80 A, when connected across a three-phase, 1100 V, 50 Hz supply. Find the circuit constants of the load per phase.

Solution

$$P = 100 \text{ kW}$$
$$I_L = 80 \text{ A}$$
$$V_L = 1100 \text{ V}$$
$$f = 50 \text{ Hz}$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{1100}{\sqrt{3}} = 635.08 \text{ V}$$
$$I_{ph} = I_L = 80 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{635.08}{80} = 7.94 \ \Omega$$
$$P = \sqrt{3} \, V_L \, I_L \cos \phi$$
$$100 \times 10^3 = \sqrt{3} \times 1100 \times 80 \times \cos \phi$$
$$\cos \phi = 0.656 \text{ (leading)}$$
$$\phi = 49°$$
$$R_{ph} = Z_{ph} \cos \phi = 7.94 \times 0.656 = 5.21 \ \Omega$$
$$X_{ph} = Z_{ph} \sin \phi = 7.94 \times \sin (49°) = 6 \ \Omega$$

This reactance will be capacitive in nature as the current is leading.

$$X_C = \frac{1}{2\pi fC}$$
$$6 = \frac{1}{2\pi \times 50 \times C}$$
$$C = 530.52 \ \mu\text{F}$$

Example 21

Three identical impedances are connected in delta to a three-phase supply of 400 V. The line current is 34.65 A, and the total power taken from the supply is 14.4 kW. Calculate the resistance and reactance values of each impedance.

Solution

$$V_L = 400 \text{ V}$$
$$I_L = 34.65 \text{ A}$$

$$P = 14.4 \text{ kW}$$

For a delta-connected load,

$$V_L = V_{ph} = 400 \text{ V}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{34.65}{\sqrt{3}} = 20 \text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{400}{20} = 20\ \Omega$$

$$P = \sqrt{3}\ V_L I_L \cos\phi$$

$$14.4 \times 10^3 = \sqrt{3} \times 400 \times 34.65 \times \cos\phi$$

$$\cos\phi = 0.6$$

$$\phi = 53.13°$$

$$R_{ph} = Z_{ph}\cos\phi = 20 \times 0.6 = 12\ \Omega$$

$$X_{ph} = Z_{ph}\sin\phi = 20 \times \sin(53.13°) = 16\ \Omega$$

Example 22

Three similar coils, connected in star, take a total power of 18 kW at a power factor of 0.866 lagging from a three-phase, 400-volt, 50 Hz system. Calculate the resistance and inductance of each coil. **[May 2014]**

Solution

$$P = 18 \text{ kW}$$

$$\text{pf} = 0.866 \text{ (lagging)}$$

$$V_L = 400 \text{ V}$$

$$f = 50 \text{ Hz}$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$P = \sqrt{3}\ V_L\ I_L \cos\phi$$

$$18 \times 10^3 = \sqrt{3} \times 400 \times I_L \times 0.866$$

$$I_L = 30 \text{ A}$$

$$I_{ph} = I_L = 30 \text{ A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{30} = 7.7\ \Omega$$

$$\phi = \cos^{-1}(0.866) = 30°$$

$$\bar{Z}_{ph} = Z_{ph} \angle \phi = 7.7 \angle 30° = 6.67 + j3.85\ \Omega$$

$$R_{ph} = 6.67\ \Omega$$

$$X_{ph} = 3.85\ \Omega$$

$$X_{ph} = 2\pi f L$$

$$3.85 = 2\pi \times 50 \times L$$

$$L = 12.25\ \text{mH}$$

Example 23

A balanced three-phase load connected in delta, draws a power of 10 kW at 440 V at a pf of 0.6 lead, find the values of circuit elements and reactive volt-amperes drawn. **[May 2016]**

Solution $P = 10$ kW

$$V_L = 440\ \text{V}$$

$$\text{pf} = 0.6\ (\text{lead})$$

For a delta-connected load,

(i) Values of circuit elements

$$V_L = V_{ph} = 440\ \text{V}$$

$$P = \sqrt{3}\ V_L I_L \cos\phi$$

$$10 \times 10^3 = \sqrt{3} \times 440 \times I_L \times 0.6$$

$$I_L = 21.87\ \text{A}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{21.87}{\sqrt{3}} = 12.63\ \text{A}$$

$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{440}{12.63} = 34.84\ \Omega$$

$$\phi = \cos^{-1}(0.6) = 53.13°$$

$$R_{ph} = Z_{ph}\cos\phi = 34.84 \times 0.6 = 20.90\ \Omega$$

$$X_{ph} = Z_{ph}\sin\phi = 34.84 \times 0.8 = 27.87\ \Omega$$

(ii) Reactive volt-amperes drawn

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 440 \times 21.87 \times 0.8 = 13.33\ \text{kVAR}$$

Example 24

Find the values of circuit elements and reactive volt-ampere drawn for a balanced 3-phase load connected in delta and drawing a power of 12 kW at 440 V. The power factor is 0.7 leading. **[Dec 2013]**

Solution

$$P = 12 \text{ kW}$$
$$V_L = 440 \text{ V}$$
$$\text{pf} = 0.7 \text{ (leading)}$$

For a delta-connected load,

(i) Values of circuit elements

$$V_L = V_{ph} = 440 \text{ V}$$
$$P = \sqrt{3} V_L I_L \cos\phi$$
$$12 \times 10^3 = \sqrt{3} \times 440 \times I_L \times 0.7$$
$$I_L = 22.49 \text{ A}$$
$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{22.49}{\sqrt{3}} = 12.98 \text{ A}$$
$$Z_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{440}{12.98} = 33.9\ \Omega$$
$$R_{ph} = Z_{ph} \cos\phi = 33.9 \times 0.7 = 23.73\ \Omega$$
$$X_{ph} = Z_{ph} \sin\phi = 33.9 \times \sin(\cos^{-1} 0.7) = 33.9 \times 0.71 = 24.07\ \Omega$$

(ii) Reactive volt-amperes drawn

$$Q = \sqrt{3} V_L I_L \sin\phi = \sqrt{3} \times 440 \times 22.49 \times 0.71 = 12.17 \text{ kVAR}$$

Example 25

Each leg of a balanced, delta-connected load consists of a 7 Ω resistance in series with a 4 Ω inductive reactance. The line-to-line voltages are

$$E_{ab} = 2360 \angle 0° \text{ V}$$
$$E_{bc} = 2360 \angle -120° \text{ V}$$
$$E_{ca} = 2360 \angle 120° \text{ V}$$

Determine (i) phase current I_{ab}, I_{bc} and I_{ca} (both magnitude and phase)

(ii) each line current and its associated phase angle

(iii) the load power factor

Solution

$$R = 7\ \Omega$$
$$X_L = 4\ \Omega$$
$$V_L = 2360 \text{ V}$$

For a delta-connected load,

(i) Phase current

$$V_{ph} = V_L = 2360 \text{ V}$$

$$\bar{Z}_{ph} = 7 + j4 = 8.06 \angle 29.74° \ \Omega$$

$$\bar{I}_{ab} = \frac{\bar{E}_{ab}}{\bar{Z}_{ph}} = \frac{2360\angle 0°}{8.06\angle 29.74°} = 292.8 \angle -29.74° \text{ A}$$

$$\bar{I}_{bc} = \frac{\bar{E}_{bc}}{\bar{Z}_{ph}} = \frac{2360\angle -120°}{8.06\angle 29.74°} = 292.8 \angle -149.71° \text{ A}$$

$$\bar{I}_{ca} = \frac{\bar{E}_{ca}}{\bar{Z}_{ph}} = \frac{2360\angle 120°}{8.06\angle 29.74°} = 292.8 \angle 90.26° \text{ A}$$

(ii) Line current

In a delta-connected, three-phase system, line currents lag behind respective phase currents by 30°.

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 292.8 = 507.14 \text{ A}$$

$$\bar{I}_{La} = 507.14 \angle -59.71° \text{ A}$$

$$\bar{I}_{Lb} = 507.14 \angle -179.71° \text{ A}$$

$$\bar{I}_{Lc} = 507.14 \angle 60.26° \text{ A}$$

(iii) Load power factor

$$\text{pf} = \cos(29.74°) = 0.868 \text{ (lagging)}$$

Example 26

A three-phase, 200 kW, 50 Hz, delta-connected induction motor is supplied from a three-phase, 440 V, 50 Hz supply system. The efficiency and power factor of the three-phase induction motor are 91% and 0.86 respectively. Calculate (i) line currents, (ii) currents in each phase of the motor, (iii) active, and (iv) reactive components of phase current.

Solution

$$P_o = 200 \text{ kW}$$

$$V_L = 440 \text{ V}$$

$$f = 50 \text{ Hz}$$

$$\eta = 91\%$$

$$\text{pf} = 0.86$$

For a delta-connected load (induction motor),

(i) Line current

$$\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{P_o}{P_i}$$

$$0.91 = \frac{200 \times 10^3}{P_i}$$

$$P_i = 219.78 \text{ kW}$$

$$P_i = \sqrt{3}\, V_L\, I_L \cos\phi$$

$$219.78 \times 10^3 = \sqrt{3} \times 440 \times I_L \times 0.86$$

$$I_L = 335.3 \text{ A}$$

(ii) Currents in each phase of motor

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{335.3}{\sqrt{3}} = 193.6 \text{ A}$$

(iii) Active component of phase current

$$I_{ph} \cos\phi = 193.6 \times 0.86 = 166.5 \text{ A}$$

(iv) Reactive component of phase current

$$I_{ph} \sin\phi = 193.6 \times \sin(\cos^{-1} 0.86) = 193.6 \times 0.51 = 98.7 \text{ A}$$

Example 27

A three-phase, 400 V, star-connected alternator supplies a three-phase, 112 kW, mesh-connected induction motor of efficiency and power factor 0.88 and 0.86 respectively. Find the (i) current in each motor phase, (ii) current in each alternator phase, and (iii) active and reactive components of current in each case.

Solution

$$V_L = 400 \text{ V}$$

$$P_o = 112 \text{ kW}$$

$$\eta = 0.88$$

$$\text{pf} = 0.86$$

For a mesh-connected load (induction motor),

(i) Current in each motor phase

$$V_{ph} = V_L = 400 \text{ V}$$

$$\eta = \frac{\text{Output power}}{\text{Input power}} = \frac{P_o}{P_i}$$

$$0.88 = \frac{112 \times 10^3}{P_i}$$

$$P_i = 127.27 \text{ kW}$$

$$P_i = \sqrt{3}\, V_L\, I_L \cos\phi$$

$$127.27 \times 10^3 = \sqrt{3} \times 400 \times I_L \times 0.86$$

$$I_L = 213.6 \text{ A}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{213.6}{\sqrt{3}} = 123.32 \text{ A}$$

Current in a star-connected alternator phase will be same as the line current drawn by the motor.

(ii) Current in each alternator phase

$$I_L = 213.6 \text{ A}$$

(iii) Active component of current in each phase of motor

$$I_{ph} \cos \phi = 123.32 \times 0.86 = 105.06 \text{ A}$$

Reactive component of current in each phase of the motor

$$I_{ph} \sin \phi = 123.32 \times \sin (\cos^{-1} 0.86) = 123.32 \times 0.51 = 62.89 \text{ A}$$

(iv) Active component of current in each alternator phase

$$I_L \cos \phi = 213.6 \times 0.86 = 183.7 \text{ A}$$

Reactive component of current in each alternator phase

$$I_L \sin \phi = 213.6 \times \sin (\cos^{-1} 0.86) = 213.6 \times 0.51 = 108.94 \text{ A}$$

Example 28

Three similar resistors are connected in star across 400 V, three-phase lines. The line current is 5 A. Calculate the value of each resistor. To what value should the line voltage be changed to obtain the same line current with the resistors connected in delta?

Solution

$$V_L = 400 \text{ V}$$

$$I_L = 5 \text{ A}$$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = I_L = 5 \text{ A}$$

$$Z_{ph} = R_{ph} = \frac{V_{ph}}{I_{ph}} = \frac{230.94}{5} = 46.19 \ \Omega$$

For a delta-connected load,

$$I_L = 5 \text{ A}$$

$$R_{ph} = 46.19 \ \Omega$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = \frac{5}{\sqrt{3}} \text{ A}$$

$$V_{ph} = I_{ph} R_{ph} = \frac{5}{\sqrt{3}} \times 46.19 = 133.33 \text{ V}$$

$$V_L = 133.33 \text{ V}$$

Voltage needed is one-third of the star value.

Example 29

Three 100 Ω, non-inductive resistors are connected in (a) star, and (b) delta across a 400 V, 50 Hz, three-phase supply. Calculate the power taken from the supply in each case. If one of the resistors is open circuited, what would be the value of total power taken from the mains in each of the two cases?

Solution $V_L = 400 \text{ V}$

$Z_{ph} = 100\ \Omega$

For a star-connected load,

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{100} = 2.31 \text{ A}$$

$$I_L = I_{ph} = 2.31 \text{ A}$$

$$\cos\phi = 1 \qquad \text{(For pure resistor, pf = 1)}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 2.31 \times 1 = 1600.41 \text{ W}$$

For a delta-connected load,

$$V_{ph} = V_L = 400 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{400}{100} = 4 \text{ A}$$

$$I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 4 = 6.93 \text{ A}$$

$$P = \sqrt{3}\, V_L I_L \cos\phi = \sqrt{3} \times 400 \times 6.93 \times 1 = 4801.24 \text{ W}$$

When one of the resistors is open circuited

(i) Star connection The circuit consists of two 100 Ω resistors in series across a 400 V supply.

$$\text{Currents in lines } A \text{ and } C = \frac{400}{200} = 2 \text{ A}$$

$$\text{Power taken from the mains} = 400 \times 2 = 800 \text{ W}$$

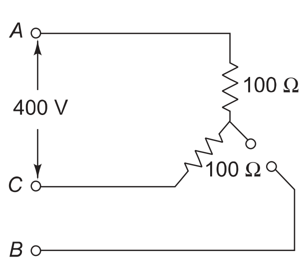


Fig. 5.22(a) *Star connection*

Hence, when one of the resistors is open circuited, the power consumption is reduced by half.

(ii) Delta connection In this case, currents in A and C remain as usual 120° out of phase with each other.

$$\text{Current in each phase} = \frac{400}{100} = 4 \text{ A}$$

Power taken from the mains $= 2 \times 4 \times 400 = 3200$ W

Hence, when one of the resistors is open circuited, the power consumption is reduced by one-third.

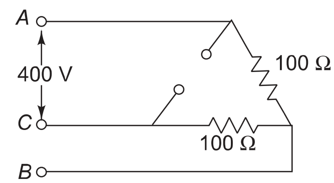


Fig. 5.22(b) *Delta connection*

Example 30

Three identical impedances of 10 ∠30° Ω each are connected in star and another set of three identical impedances of 18 ∠60° Ω are connected in delta. If both the sets of impedances are connected across a balanced, three-phase 400 V supply, find the line current, total volt-amperes, active power and reactive power.

Solution

$$\bar{Z}_Y = 10 \angle 30° \ \Omega$$
$$\bar{Z}_\Delta = 18 \angle 60° \ \Omega$$
$$V_L = 400 \text{ V}$$

Three identical delta impedances can be converted into equivalent star impedances.

$$\bar{Z}'_Y = \frac{\bar{Z}_\Delta}{3} = \frac{18\angle 60°}{3} = 6\angle 60° \ \Omega$$

Now two star-connected impedances of 10 ∠30° Ω and 6 ∠60° Ω are connected in parallel across a three-phase supply.

$$\bar{Z}_{eq} = \frac{(10\angle 30°)(6\angle 60°)}{10\angle 30° + 6\angle 60°} = 3.87 \angle 48.83° \ \Omega$$

For a star-connected load,

(i) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_{ph}}{Z_{eq}} = \frac{230.94}{3.87} = 59.67 \text{ A}$$

$$I_L = I_{ph} = 59.67 \text{ A}$$

(ii) Total volt-amperes

$$S = \sqrt{3} \ V_L I_L = \sqrt{3} \times 400 \times 59.67 = 41.34 \text{ kVA}$$

(iii) Active power

$$P = \sqrt{3} \ V_L I_L \cos\phi = \sqrt{3} \times 400 \times 59.67 \times \cos(48.83°) = 27.21 \text{ kW}$$

(iv) Reactive power

$$Q = \sqrt{3} \ V_L I_L \sin\phi = \sqrt{3} \times 400 \times 59.67 \times \sin(48.83°) = 31.12 \text{ kVAR}$$

Example 31

Three star-connected impedances $Z_Y = (20 + j37.7)\ \Omega$ per phase are connected in parallel with three delta-connected impedances $Z_\Delta = (30 - j159.3)\ \Omega$ per phase. The line voltage is 398 V. Find the line current, pf, active and reactive power taken by the combination.

Solution

$$\bar{Z}_Y = 20 + j37.7 = 42.68\ \angle 62.05°\ \Omega$$
$$\bar{Z}_\Delta = 30 - j159.3 = 162.1\ \angle -79.3°\ \Omega$$
$$V_L = 398\ \text{V}$$

Three identical delta-connected impedances can be converted into equivalent star impedances.

$$\bar{Z}'_Y = \frac{162.1\angle -79.3°}{3} = 54.03\ \angle -79.3°\ \Omega$$

Now two star-connected impedances of 42.68 $\angle 62.05°\ \Omega$ and 54.03 $\angle -79.3°\ \Omega$ are connected in parallel across the three-phase supply.

$$\bar{Z}_{\text{eq}} = \frac{(42.68\angle 62.05°)(54.03\angle -79.3°)}{42.68\angle 62.05° + 54.03\angle -79.3°)} = 68.33\angle 9.88°\ \Omega$$

For a star-connected load,

(i) Line current

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{398}{\sqrt{3}} = 229.79\ \text{V}$$
$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{V_{ph}}{Z_{eq}} = \frac{229.79}{68.33} = 3.36\ \text{A}$$
$$I_L = I_{ph} = 3.36\ \text{A}$$

(ii) Power factor

$$\text{pf} = \cos\phi = \cos(9.88°) = 0.99\ \text{(lagging)}$$

(iii) Active power

$$P = \sqrt{3}\ V_L I_L \cos\phi = \sqrt{3} \times 398 \times 3.36 \times 0.99 = 2.29\ \text{kW}$$

(iv) Reactive power

$$Q = \sqrt{3}\ V_L I_L \sin\phi = \sqrt{3} \times 398 \times 3.36 \times \sin(9.88°) = 397.43\ \text{VAR}$$

Example 32

Three coils, each having a resistance of 20 Ω and a reactance of 15 Ω, are connected in star to a 400 V, three-phase, 50 Hz supply. Calculate (i) line current, (ii) power supplied, and (iii) power factor. If three capacitors, each of same capacitance, are connected in delta to the same supply so as to form a parallel circuit with the above coils, calculate the capacitance of each capacitor to obtain a resultant power factor of 0.95 lagging.

Solution

$$R_{ph} = 20\ \Omega$$
$$X_{ph} = 15\ \Omega$$
$$V_L = 400\ \text{V}$$

For a star-connected load,

(i) Line current

$$\bar{Z}_{ph} = R_{ph} + jX_{ph} = 20 + j15 = 25 \angle 36.87° \ \Omega$$

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.94 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{230.94}{25} = 9.24 \text{ A}$$

$$I_L = I_{ph} = 9.24 \text{ A}$$

(ii) Power supplied

$$P_1 = \sqrt{3}\, V_L I_L \cos \phi_1 = \sqrt{3} \times 400 \times 9.24 \times \cos (36.87°) = 5.12 \text{ kW}$$

(iii) Power factor

$$\text{pf} = \cos \phi_1 = \cos (36.87°) = 0.8 \text{ (lagging)}$$

(iv) Value of capacitance of each capacitor

$$Q_1 = \sqrt{3}\, V_L I_L \sin \phi_1 = \sqrt{3} \times 400 \times 9.24 \times \sin (36.87°) = 3.84 \text{ kVAR}$$

When capacitors are connected in delta to the same supply

$$\text{pf} = 0.95$$

$$\phi_2 = \cos^{-1} (0.95) = 18.19°$$

$$\tan \phi_2 = \tan (18.19°) = 0.33$$

Since capacitors do not absorb any power, power remains the same even when capacitors are connected. But reactive power changes.

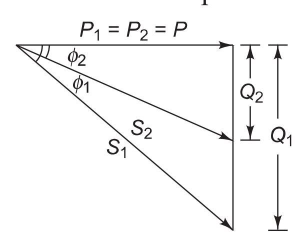


Fig. 5.23

$$P_2 = 5.12 \text{ kW}$$

$$Q_2 = P_2 \tan \phi_2 = 5.12 \times 0.33 = 1.69 \text{ kVAR}$$

Difference in reactive power is supplied by three capacitors.

$$Q = Q_1 - Q_2 = 3.84 - 1.69 = 2.15 \text{ kVAR}$$

$$Q = \sqrt{3}\, V_L I_L \sin \phi$$

$$2.15 \times 10^3 = \sqrt{3} \times 400 \times I_L \times \sin (90°)$$

$$I_L = 3.1 \text{ A}$$

$$I_{ph} = \frac{I_L}{\sqrt{3}} = 1.79 \text{ A}$$

$$I_{ph} = \frac{V_{ph}}{X_C} = V_{ph} \times 2\pi f C$$

$$C = \frac{I_{ph}}{V_{ph} \times 2\pi f} = \frac{1.79}{400 \times 2\pi \times 50} = 14.24 \ \mu\text{F}$$

CHAPTER 12

Polyphase Circuits

KEY CONCEPTS

- Single-Phase Power Systems
- Three-Phase Power Systems
- Three-Phase Sources
- Line Versus Phase Voltage
- Line Versus Phase Current
- Y-Connected Networks
- Δ-Connected Networks
- Balanced Loads
- Per-Phase Analysis
- Power Measurement in Three-Phase Systems

INTRODUCTION

The vast majority of power is supplied to consumers in the form of sinusoidal voltages and currents, typically referred to as alternating current or simply *ac*. Although there are exceptions, for example, some types of train motors, most equipment is designed to run on either 50 or 60 Hz. Most 60 Hz systems are now standardized to run on 120 V, whereas 50 Hz systems typically correspond to 240 V (both voltages being quoted in rms units). The actual voltage delivered to an appliance can vary somewhat from these values, and distribution systems employ significantly higher voltages to minimize the current and hence cable size. Originally Thomas Edison advocated a purely dc power distribution network, purportedly due to his preference for the simple algebra required to analyze such circuits. Nikola Tesla and George Westinghouse, two other pioneers in the field of electricity, proposed ac distribution systems as the achievable losses were significantly lower. Ultimately they were more persuasive, despite some rather theatrical demonstrations on the part of Edison.

The transient response of ac power systems is of interest when determining the peak power demand, since most equipment requires more current to start up than it does to run continuously. Often, however, it is the steady-state operation that is of primary interest, so our experience with phasor-based analysis will prove to be handy. In this chapter we introduce a new type of voltage source, the three-phase source, which can be connected in either a three- or four-wire Y configuration or a three-wire Δ configuration. Loads can also be either Y- or Δ-connected, depending on the application.

12.1 POLYPHASE SYSTEMS

So far, whenever we used the term "sinusoidal source" we pictured a single sinusoidal voltage or current having a particular amplitude, frequency, and phase. In this chapter, we introduce the concept of ***polyphase*** sources, focusing on three-phase systems in particular. There are distinct advantages in using rotating machinery to generate three-phase power rather than single-phase power, and there are economical advantages in favor of the transmission of power in a three-phase system. Although most of the electrical equipment we have encountered so far is single-phase, three-phase equipment is not uncommon, especially in manufacturing environments. In particular, motors used in large refrigeration systems and in machining facilities are often wired for three-phase power. For the remaining applications, once we have become familiar with the basics of polyphase systems, we will find that it is simple to obtain single-phase power by just connecting to a single "leg" of a polyphase system.

Let us look briefly at the most common polyphase system, a balanced three-phase system. The source has three terminals (not counting a ***neutral*** or ***ground*** connection), and voltmeter measurements will show that sinusoidal voltages of equal amplitude are present between any two terminals. However, these voltages are not in phase; each of the three voltages is 120° out of phase with each of the other two, the sign of the phase angle depending on the sense of the voltages. One possible set of voltages is shown in Fig. 12.1. A ***balanced load*** draws power equally from all three phases. *At no instant does the instantaneous power drawn by the total load reach zero; in fact, the total instantaneous power is constant*. This is an advantage in rotating machinery, for it keeps the torque on the rotor much more constant than it would be if a single-phase source were used. As a result, there is less vibration.

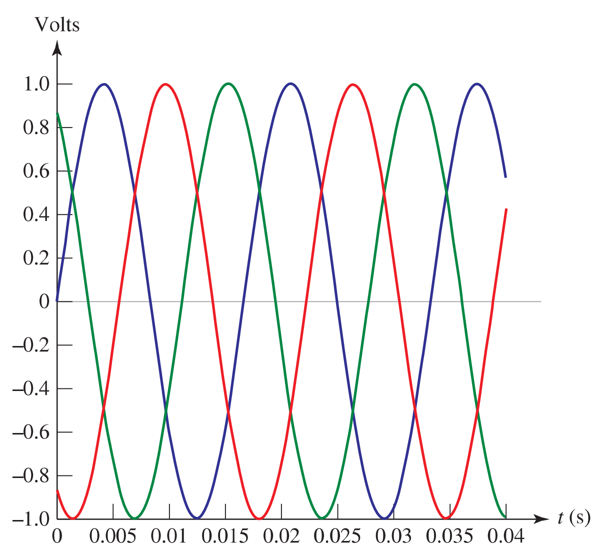
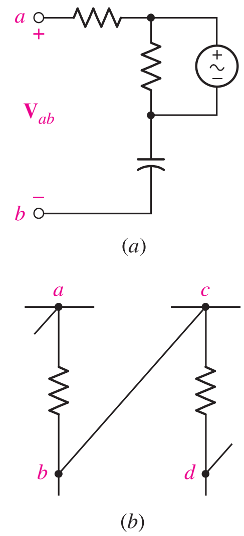


FIGURE 12.1 An example set of three voltages, each of which is 120° out of phase with the other two. As can be seen, only one of the voltages is zero at any particular instant.

The use of a higher number of phases, such as 6- and 12-phase systems, is limited almost entirely to the supply of power to large ***rectifiers.*** Rectifiers convert alternating current to direct current by only allowing current to flow to the load in one direction, so that the sign of the voltage across the load remains the same. The rectifier output is a direct current plus a smaller pulsating component, or ripple, which decreases as the number of phases increases.

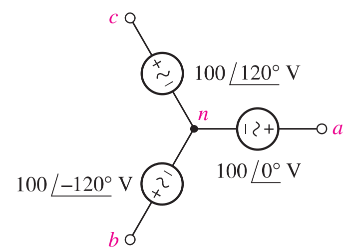
Almost without exception, polyphase systems in practice contain sources which may be closely approximated by ideal voltage sources or by ideal voltage sources in series with small internal impedances. Three-phase current sources are extremely rare.



■ **FIGURE 12.2** (*a*) The definition of the voltage $\mathbf{V}_{ab}$. (*b*) $\mathbf{V}_{ad} = \mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{cd} = \mathbf{V}_{ab} + \mathbf{V}_{cd}$.

Double-Subscript Notation

It is convenient to describe polyphase voltages and currents using ***double-subscript notation.*** With this notation, a voltage or current, such as $\mathbf{V}_{ab}$ or $\mathbf{I}_{aA}$, has more meaning than if it were indicated simply as $\mathbf{V}_3$ or $\mathbf{I}_x$. By definition, the voltage of point *a* with respect to point *b* is $\mathbf{V}_{ab}$. Thus, the plus sign is located at *a*, as indicated in Fig. 12.2*a*. We therefore consider the double subscripts to be *equivalent* to a plus-minus sign pair; the use of both would be redundant. With reference to Fig. 12.2*b*, for example, we see that $\mathbf{V}_{ad} = \mathbf{V}_{ab} + \mathbf{V}_{cd}$. The advantage of the double-subscript notation lies in the fact that Kirchhoff's voltage law requires the voltage between two points to be the same, regardless of the path chosen between the points; thus $\mathbf{V}_{ad} = \mathbf{V}_{ab} + \mathbf{V}_{bd} = \mathbf{V}_{ac} + \mathbf{V}_{cd} = \mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{cd}$, and so forth. The benefit of this is that KVL may be satisfied without reference to the circuit diagram; correct equations may be written even though a point, or subscript letter, is included which is not marked on the diagram. For example, we might have written $\mathbf{V}_{ad} = \mathbf{V}_{ax} + \mathbf{V}_{xd}$, where *x* identifies the location of any interesting point of our choice.



■ **FIGURE 12.3** A network used as a numerical example of double-subscript voltage notation.

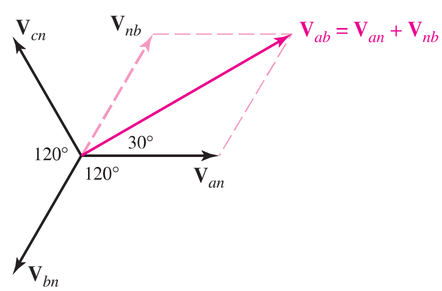
One possible representation of a three-phase system of voltages[1] is shown in Fig. 12.3. Let us assume that the voltages $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are known:

$$\mathbf{V}_{an} = 100\underline{/0^\circ} \text{ V}$$
$$\mathbf{V}_{bn} = 100\underline{/-120^\circ} \text{ V}$$
$$\mathbf{V}_{cn} = 100\underline{/-240^\circ} \text{ V}$$

The voltage $\mathbf{V}_{ab}$ may be found, with an eye on the subscripts, as

$$\begin{aligned}\mathbf{V}_{ab} &= \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} \\ &= 100\underline{/0^\circ} - 100\underline{/-120^\circ} \text{ V} \\ &= 100 - (-50 - j86.6) \text{ V} \\ &= 173.2\underline{/30^\circ} \text{ V}\end{aligned}$$

The three given voltages and the construction of the phasor $\mathbf{V}_{ab}$ are shown on the phasor diagram of Fig. 12.4.



■ **FIGURE 12.4** This phasor diagram illustrates the graphical use of the double-subscript voltage convention to obtain $\mathbf{V}_{ab}$ for the network of Fig. 12.3.

A double-subscript notation may also be applied to currents. We define the current $\mathbf{I}_{ab}$ as the current flowing from *a* to *b by the most direct path*. In

(1) In keeping with power industry convention, rms values for currents and voltages will be used *implicitly* throughout this chapter.

every complete circuit we consider, there must of course be at least two possible paths between the points a and b, and we agree that we will not use double-subscript notation unless it is obvious that one path is much shorter, or much more direct. Usually this path is through a single element. Thus, the current $\mathbf{I}_{ab}$ is correctly indicated in Fig. 12.5. In fact, we do not even need the direction arrow when talking about this current; the subscripts *tell* us the direction. However, the identification of a current as $\mathbf{I}_{cd}$ for the circuit of Fig. 12.5 would cause confusion.

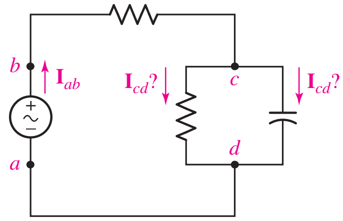


FIGURE 12.5 An illustration of the use and *misuse* of the double-subscript convention for current notation.

PRACTICE

12.1 Let $\mathbf{V}_{ab} = 100\underline{/0^\circ}$ V, $\mathbf{V}_{bd} = 40\underline{/80^\circ}$ V, and $\mathbf{V}_{ca} = 70\underline{/200^\circ}$ V. Find (*a*) $\mathbf{V}_{ad}$; (*b*) $\mathbf{V}_{bc}$; (*c*) $\mathbf{V}_{cd}$.

12.2 Refer to the circuit of Fig. 12.6 and let $\mathbf{I}_{fj} = 3$ A, $\mathbf{I}_{de} = 2$ A, and $\mathbf{I}_{hd} = -6$ A. Find (*a*) $\mathbf{I}_{cd}$; (*b*) $\mathbf{I}_{ef}$; (*c*) $\mathbf{I}_{ij}$.

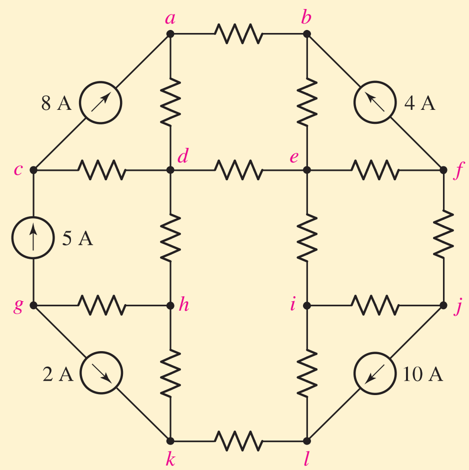


FIGURE 12.6

Ans: 12.1: $114.0\underline{/20.2^\circ}$ V; $41.8\underline{/145.0^\circ}$ V; $44.0\underline{/20.6^\circ}$ V. 12.2: −3 A; 7 A; 7 A.

12.2 SINGLE-PHASE THREE-WIRE SYSTEMS

Before studying polyphase systems in detail, it can be helpful first to look at a simple single-phase three-wire system. A *single-phase three-wire source* is defined as a source having three output terminals, such as a, n, and b in Fig. 12.7*a*, at which the phasor voltages $\mathbf{V}_{an}$ and $\mathbf{V}_{nb}$ are equal. The source may therefore be represented by the combination of two identical voltage sources; in Fig. 12.7*b*, $\mathbf{V}_{an} = \mathbf{V}_{nb} = \mathbf{V}_1$. It is apparent that $\mathbf{V}_{ab} = 2\mathbf{V}_{an} = 2\mathbf{V}_{nb}$, and we therefore have a source to which loads operating at either of two voltages may be connected. The normal North American household system is single-phase three-wire, permitting the operation of both 110 V and 220 V appliances. The higher-voltage appliances are normally those drawing larger amounts of power; operation at higher voltage results in a smaller current draw for the same power. Smaller-diameter wire may consequently be used safely in the appliance, the household distribution

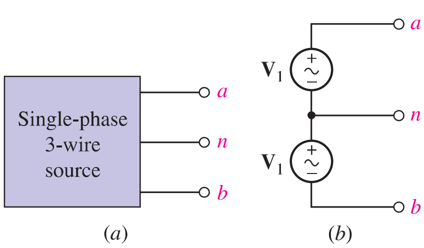


FIGURE 12.7 (*a*) A single-phase three-wire source. (*b*) The representation of a single-phase three-wire source by two identical voltage sources.

system, and the distribution system of the utility company, as larger-diameter wire must be used with higher currents to reduce the heat produced due to the resistance of the wire.

The name ***single-phase*** arises because the voltages $\mathbf{V}_{an}$ and $\mathbf{V}_{nb}$, being equal, must have the same phase angle. From another viewpoint, however, the voltages between the outer wires and the central wire, which is usually referred to as the *neutral,* are exactly 180° out of phase. That is, $\mathbf{V}_{an} = -\mathbf{V}_{bn}$, and $\mathbf{V}_{an} + \mathbf{V}_{bn} = 0$. Later, we will see that balanced polyphase systems are characterized by a set of voltages of equal *amplitude* whose (phasor) sum is zero. From this viewpoint, then, the single-phase three-wire system is really a balanced two-phase system. *Two-phase,* however, is a term that is traditionally reserved for a relatively unimportant unbalanced system utilizing two voltage sources 90° out of phase.

Let us now consider a single-phase three-wire system that contains identical loads $\mathbf{Z}_p$ between each outer wire and the neutral (Fig. 12.8). We first assume that the wires connecting the source to the load are perfect conductors. Since

$$\mathbf{V}_{an} = \mathbf{V}_{nb}$$

then,

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{an}}{\mathbf{Z}_p} = \mathbf{I}_{Bb} = \frac{\mathbf{V}_{nb}}{\mathbf{Z}_p}$$

and therefore

$$\mathbf{I}_{nN} = \mathbf{I}_{Bb} + \mathbf{I}_{Aa} = \mathbf{I}_{Bb} - \mathbf{I}_{aA} = 0$$

Thus there is no current in the neutral wire, and it could be removed without changing any current or voltage in the system. This result is achieved through the equality of the two loads and of the two sources.

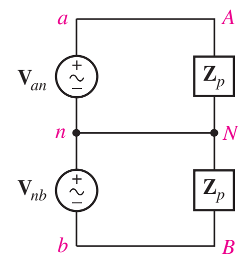


FIGURE 12.8 A simple single-phase three-wire system. The two loads are identical, and the neutral current is zero.

Effect of Finite Wire Impedance

We next consider the effect of a finite impedance in each of the wires. If lines aA and bB each have the same impedance, this impedance may be added to $\mathbf{Z}_p$, resulting in two equal loads once more, and zero neutral current. Now let us allow the neutral wire to possess some impedance $\mathbf{Z}_n$. Without carrying out any detailed analysis, superposition should show us that the symmetry of the circuit will still cause zero neutral current. Moreover, the addition of any impedance connected directly from one of the outer lines to the other outer line also yields a symmetrical circuit and zero neutral current. Thus, zero neutral current is a consequence of a balanced, or symmetrical, load; nonzero impedance in the neutral wire does not destroy the symmetry.

The most general single-phase three-wire system will contain unequal loads between each outside line and the neutral and another load directly between the two outer lines; the impedances of the two outer lines may be expected to be approximately equal, but the neutral impedance is often slightly larger. Let us consider an example of such a system, with particular interest in the current that may flow now through the neutral wire, as well as the overall efficiency with which our system is transmitting power to the unbalanced load.

EXAMPLE 12.1

Analyze the system shown in Fig. 12.9 and determine the power delivered to each of the three loads as well as the power lost in the neutral wire and each of the two lines.

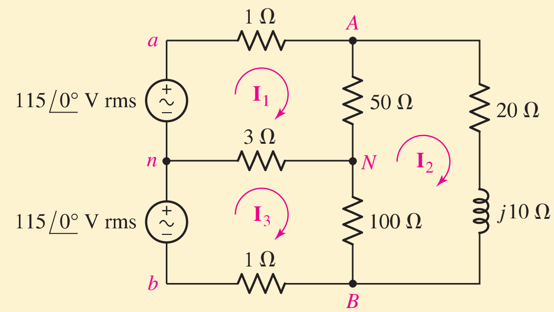


FIGURE 12.9 A typical single-phase three-wire system.

Identify the goal of the problem.

The three loads in the circuit are the 50 Ω resistor, the 100 Ω resistor, and a $20 + j10\ \Omega$ impedance. Each of the two lines has a resistance of 1 Ω, and the neutral wire has a resistance of 3 Ω. We need the current through each of these in order to determine power.

Collect the known information.

We have a single-phase three-wire system; the circuit diagram of Fig. 12.9 is completely labeled. The computed currents will be in rms units.

Devise a plan.

The circuit is conducive to mesh analysis, having three clearly defined meshes. The result of the analysis will be a set of mesh currents, which can then be used to compute absorbed power.

Construct an appropriate set of equations.

The three mesh equations are:

$$-115\underline{/0^\circ} + \mathbf{I}_1 + 50(\mathbf{I}_1 - \mathbf{I}_2) + 3(\mathbf{I}_1 - \mathbf{I}_3) = 0$$
$$(20 + j10)\mathbf{I}_2 + 100(\mathbf{I}_2 - \mathbf{I}_3) + 50(\mathbf{I}_2 - \mathbf{I}_1) = 0$$
$$-115\underline{/0^\circ} + 3(\mathbf{I}_3 - \mathbf{I}_1) + 100(\mathbf{I}_3 - \mathbf{I}_2) + \mathbf{I}_3 = 0$$

which can be rearranged to obtain the following three equations

$$\begin{aligned} 54\mathbf{I}_1 \quad & -50\mathbf{I}_2 & -3\mathbf{I}_3 &= 115\underline{/0^\circ} \\ -50\mathbf{I}_1 \quad & +(170 + j10)\mathbf{I}_2 & -100\mathbf{I}_3 &= 0 \\ -3\mathbf{I}_1 \quad & -100\mathbf{I}_2 & +104\mathbf{I}_3 &= 115\underline{/0^\circ} \end{aligned}$$

Determine if additional information is required.

We have a set of three equations in three unknowns, so it is possible to attempt a solution at this point.

▶ *Attempt a solution.*

Solving for the phasor currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ using a scientific calculator, we find

$$\mathbf{I}_1 = 11.24\underline{/-19.83^\circ} \text{ A}$$
$$\mathbf{I}_2 = 9.389\underline{/-24.47^\circ} \text{ A}$$
$$\mathbf{I}_3 = 10.37\underline{/-21.80^\circ} \text{ A}$$

The currents in the outer lines are thus

$$\mathbf{I}_{aA} = \mathbf{I}_1 = 11.24\underline{/-19.83^\circ} \text{ A}$$

and

$$\mathbf{I}_{bB} = -\mathbf{I}_3 = 10.37\underline{/158.20^\circ} \text{ A}$$

and the smaller neutral current is

$$\mathbf{I}_{nN} = \mathbf{I}_3 - \mathbf{I}_1 = 0.9459\underline{/-177.7^\circ} \text{ A}$$

The average power drawn by each load may thus be determined:

$$P_{50} = |\mathbf{I}_1 - \mathbf{I}_2|^2 (50) = 206 \text{ W}$$
$$P_{100} = |\mathbf{I}_3 - \mathbf{I}_2|^2 (100) = 117 \text{ W}$$
$$P_{20+j10} = |\mathbf{I}_2|^2 (20) = 1763 \text{ W}$$

Note that we do not need to include a factor of $\frac{1}{2}$ since we are working with rms current values.

The total load power is 2086 W. The loss in each of the wires is next found:

$$P_{aA} = |\mathbf{I}_1|^2 (1) = 126 \text{ W}$$
$$P_{bB} = |\mathbf{I}_3|^2 (1) = 108 \text{ W}$$
$$P_{nN} = |\mathbf{I}_{nN}|^2 (3) = 3 \text{ W}$$

giving a total line loss of 237 W. The wires are evidently quite long; otherwise, the relatively high power loss in the two outer lines would cause a dangerous temperature rise.

Imagine the heat produced by two 100 W light bulbs! These outer wires must dissipate the same amount of power. In order to keep their temperature down, a large surface area is required.

▶ *Verify the solution. Is it reasonable or expected?*

The total absorbed power is 206 + 117 + 1763 + 237, or 2323 W, which may be checked by finding the power delivered by each voltage source:

$$P_{an} = 115(11.24)\cos 19.83^\circ = 1216 \text{ W}$$
$$P_{bn} = 115(10.37)\cos 21.80^\circ = 1107 \text{ W}$$

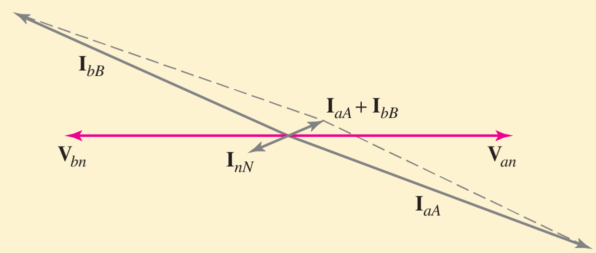
or a total of 2323 W. The ***transmission efficiency*** for the system is

$$\eta = \frac{\text{total power delivered to load}}{\text{total power generated}} = \frac{2086}{2086 + 237} = 89.8\%$$

This value would be unbelievable for a steam engine or an internal combustion engine, but it is too low for a well-designed distribution system. Larger-diameter wires should be used if the source and the load cannot be placed closer to each other.

(Continued on next page)

A phasor diagram showing the two source voltages, the currents in the outer lines, and the current in the neutral is constructed in Fig. 12.10. The fact that $\mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{nN} = 0$ is indicated on the diagram.



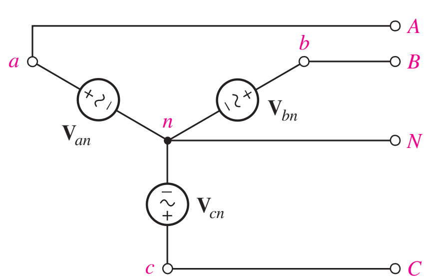
■ **FIGURE 12.10** The source voltages and three of the currents in the circuit of Fig. 12.9 are shown on a phasor diagram. Note that $\mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{nN} = 0$.

PRACTICE

12.3 Modify Fig. 12.9 by adding a 1.5 Ω resistance to each of the two outer lines, and a 2.5 Ω resistance to the neutral wire. Find the average power delivered to each of the three loads.

Ans: 153.1 W; 95.8 W; 1374 W.

12.3 THREE-PHASE Y-Y CONNECTION



■ **FIGURE 12.11** A Y-connected three-phase four-wire source.

Three-phase sources have three terminals, called the *line* terminals, and they may or may not have a fourth terminal, the *neutral* connection. We will begin by discussing a three-phase source that does have a neutral connection. It may be represented by three ideal voltage sources connected in a Y, as shown in Fig. 12.11; terminals *a*, *b*, *c*, and *n* are available. We will consider only balanced three-phase sources, which may be defined as having

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

and

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0$$

These three voltages, each existing between one line and the neutral, are called ***phase voltages.*** If we arbitrarily choose $\mathbf{V}_{an}$ as the reference, or define

$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$

where we will consistently use V_p to represent the rms *amplitude* of any of the phase voltages, then the definition of the three-phase source indicates that either

$$\mathbf{V}_{bn} = V_p\underline{/-120^\circ} \qquad \text{and} \qquad \mathbf{V}_{cn} = V_p\underline{/-240^\circ}$$

or

$$\mathbf{V}_{bn} = V_p\underline{/120^\circ} \qquad \text{and} \qquad \mathbf{V}_{cn} = V_p\underline{/240^\circ}$$

The former is called ***positive phase sequence,*** or ***abc phase sequence,*** and is shown in Fig. 12.12*a*; the latter is termed ***negative phase sequence,*** or ***cba phase sequence,*** and is indicated by the phasor diagram of Fig. 12.12*b*.

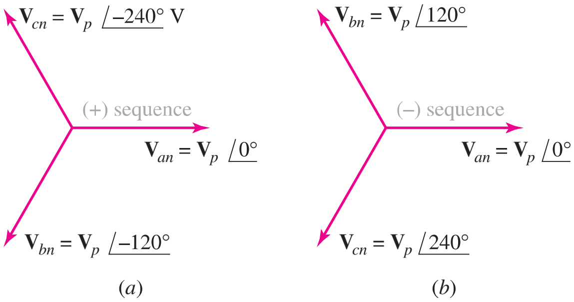


FIGURE 12.12 (*a*) Positive, or *abc*, phase sequence. (*b*) Negative, or *cba*, phase sequence.

The actual phase sequence of a physical three-phase source depends on the arbitrary choice of the three terminals to be lettered a, b, and c. They may always be chosen to provide positive phase sequence, and we will assume that this has been done in most of the systems we consider.

Line-to-Line Voltages

Let us next find the line-to-line voltages (often simply called the ***line voltages***) which are present when the phase voltages are those of Fig. 12.12a. It is easiest to do this with the help of a phasor diagram, since the angles are all multiples of 30°. The necessary construction is shown in Fig. 12.13; the results are

$$\mathbf{V}_{ab} = \sqrt{3}V_p \angle 30° \qquad [1]$$

$$\mathbf{V}_{bc} = \sqrt{3}V_p \angle -90° \qquad [2]$$

$$\mathbf{V}_{ca} = \sqrt{3}V_p \angle -210° \qquad [3]$$

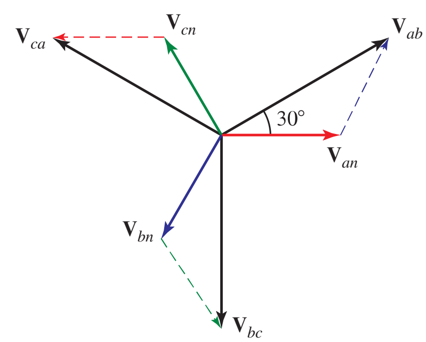


FIGURE 12.13 A phasor diagram which is used to determine the line voltages from the given phase voltages. Or, algebraically, $\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p \angle 0° - V_p \angle -120° = V_p - V_p \cos(-120°) - jV_p \sin(-120°) = V_p(1 + \frac{1}{2} + j\sqrt{3}/2) = \sqrt{3}V_p \angle 30°$.

Kirchhoff's voltage law requires the sum of these three voltages to be zero; the reader is encouraged to verify this as an exercise.

If the rms amplitude of any of the line voltages is denoted by V_L, then one of the important characteristics of the Y-connected three-phase source may be expressed as

$$\boxed{V_L = \sqrt{3}V_p}$$

Note that with positive phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$ and $\mathbf{V}_{bn}$ leads $\mathbf{V}_{cn}$, in each case by 120°, and also that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{bc}$ and $\mathbf{V}_{bc}$ leads $\mathbf{V}_{ca}$, again by 120°. The statement is true for negative phase sequence if "lags" is substituted for "leads."

Now let us connect a balanced Y-connected three-phase load to our source, using three lines and a neutral, as drawn in Fig. 12.14. The load is

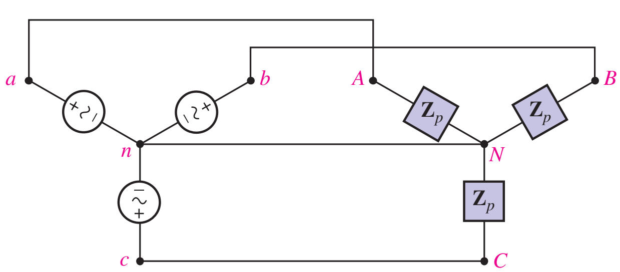


FIGURE 12.14 A balanced three-phase system, connected Y-Y and including a neutral.

represented by an impedance $\mathbf{Z}_p$ between each line and the neutral. The three line currents are found very easily, since we really have three single-phase circuits that possess one common lead:[2]

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{an}}{\mathbf{Z}_p}$$

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_p} = \frac{\mathbf{V}_{an}\underline{/-120^\circ}}{\mathbf{Z}_p} = \mathbf{I}_{aA}\underline{/-120^\circ}$$

$$\mathbf{I}_{cC} = \mathbf{I}_{aA}\underline{/-240^\circ}$$

and therefore

$$\mathbf{I}_{Nn} = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC} = 0$$

Thus, the neutral carries no current if the source and load are both balanced and if the four wires have zero impedance. How will this change if an impedance $\mathbf{Z}_L$ is inserted in series with each of the three lines and an impedance $\mathbf{Z}_n$ is inserted in the neutral? The line impedances may be combined with the three load impedances; this effective load is still balanced, and a perfectly conducting neutral wire could be removed. Thus, if no change is produced in the system with a short circuit or an open circuit between n and N, any impedance may be inserted in the neutral and the neutral current will remain zero.

It follows that, if we have balanced sources, balanced loads, and balanced line impedances, a neutral wire of any impedance may be replaced by any other impedance, including a short circuit or an open circuit; the replacement will not affect the system's voltages or currents. It is often helpful to *visualize* a short circuit between the two neutral points, whether a neutral wire is actually present or not; the problem is then reduced to three single-phase problems, all identical except for the consistent difference in phase angle. We say that we thus work the problem on a "per-phase" basis.

EXAMPLE 12.2

For the circuit of Fig. 12.15, find both the phase and line currents, and the phase and line voltages throughout the circuit; then calculate the total power dissipated in the load.

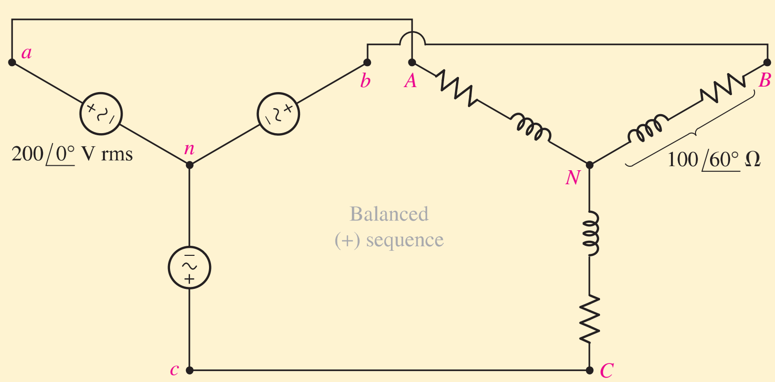


FIGURE 12.15 A balanced three-phase three-wire Y-Y connected system.

(2) This can be seen to be true by applying superposition and looking at each phase one at a time.

Since one of the source phase voltages is given and we are told to use the positive phase sequence, the three phase voltages are:

$$\mathbf{V}_{an} = 200\underline{/0^\circ}\ \text{V} \qquad \mathbf{V}_{bn} = 200\underline{/-120^\circ}\ \text{V} \qquad \mathbf{V}_{cn} = 200\underline{/-240^\circ}\ \text{V}$$

The line voltage is $200\sqrt{3} = 346$ V; the phase angle of each line voltage can be determined by constructing a phasor diagram, as we did in Fig. 12.13 (as a matter of fact, the phasor diagram of Fig. 12.13 is applicable), subtracting the phase voltages using a scientific calculator, or by invoking Eqs. [1] to [3]. We find that $\mathbf{V}_{ab}$ is $346\underline{/30^\circ}$ V, $\mathbf{V}_{bc} = 346\underline{/-90^\circ}$ V, and $\mathbf{V}_{ca} = 346\underline{/-210^\circ}$ V.

The line current for phase *A* is

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{an}}{\mathbf{Z}_p} = \frac{200\underline{/0^\circ}}{100\underline{/60^\circ}} = 2\underline{/-60^\circ}\ \text{A}$$

Since we know this is a balanced three-phase system, we may write the remaining line currents based on $\mathbf{I}_{aA}$:

$$\mathbf{I}_{bB} = 2\underline{/(-60^\circ - 120^\circ)} = 2\underline{/-180^\circ}\ \text{A}$$
$$\mathbf{I}_{cC} = 2\underline{/(-60^\circ - 240^\circ)} = 2\underline{/-300^\circ}\ \text{A}$$

Finally, the average power absorbed by phase *A* is $\text{Re}\{\mathbf{V}_{an}\mathbf{I}^*_{aA}\}$, or

$$P_{AN} = 200(2)\cos(0^\circ + 60^\circ) = 200\ \text{W}$$

Thus, the total average power drawn by the three-phase load is 600 W.

The phasor diagram for this circuit is shown in Fig. 12.16. Once we knew any of the line voltage magnitudes and any of the line current magnitudes, the angles for all three voltages and all three currents could have been obtained by simply reading the diagram.

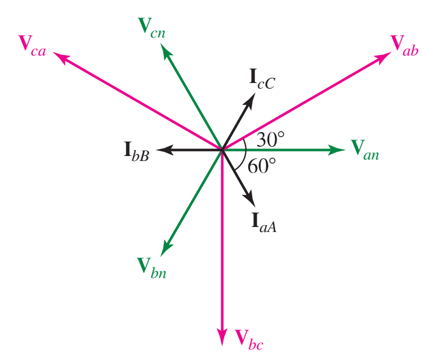


FIGURE 12.16 The phasor diagram that applies to the circuit of Fig. 12.15.

PRACTICE

12.4 A balanced three-phase three-wire system has a Y-connected load. Each phase contains three loads in parallel: $-j100\ \Omega$, $100\ \Omega$, and $50 + j50\ \Omega$. Assume positive phase sequence with $\mathbf{V}_{ab} = 400\underline{/0^\circ}$ V. Find (*a*) $\mathbf{V}_{an}$; (*b*) $\mathbf{I}_{aA}$; (*c*) the total power drawn by the load.

Ans: $231\underline{/-30^\circ}$ V; $4.62\underline{/-30^\circ}$ A; 3200 W.

Before working another example, this would be a good opportunity to quickly explore a statement made in Sec. 12.1, i.e., that even though phase voltages and currents have zero value at specific instants in time (every 1/120 s in North America), the instantaneous power delivered to the *total* load is never zero. Consider phase *A* of Example 12.2 once more, with the phase voltage and current written in the time domain:

$$v_{AN} = 200\sqrt{2}\cos(120\pi t + 0^\circ)\ \text{V}$$

The factor of $\sqrt{2}$ is required to convert from rms units.

and

$$i_{AN} = 2\sqrt{2}\cos(120\pi t - 60^\circ)\ \text{A}$$

Thus, the instantaneous power absorbed by phase A is

$$\begin{aligned} p_A(t) = v_{AN} i_{AN} &= 800 \cos(120\pi t) \cos(120\pi t - 60^\circ) \\ &= 400[\cos(-60^\circ) + \cos(240\pi t - 60^\circ)] \\ &= 200 + 400 \cos(240\pi t - 60^\circ) \text{ W} \end{aligned}$$

in a similar fashion,

$$p_B(t) = 200 + 400 \cos(240\pi t - 300^\circ) \text{ W}$$

and

$$p_C(t) = 200 + 400 \cos(240\pi t - 180^\circ) \text{ W}$$

The instantaneous power absorbed by the *total* load is therefore

$$p(t) = p_A(t) + p_B(t) + p_C(t) = 600 \text{ W}$$

independent of time, and the same value as the average power computed in Example 12.2.

EXAMPLE 12.3

A balanced three-phase system with a line voltage of 300 V is supplying a balanced Y-connected load with 1200 W at a leading PF of 0.8. Find the line current and the per-phase load impedance.

The phase voltage is $300/\sqrt{3}$ V and the per-phase power is $1200/3 = 400$ W. Thus the line current may be found from the power relationship

$$400 = \frac{300}{\sqrt{3}}(I_L)(0.8)$$

and the line current is therefore 2.89 A. The phase impedance magnitude is given by

$$|\mathbf{Z}_p| = \frac{V_p}{I_L} = \frac{300/\sqrt{3}}{2.89} = 60\ \Omega$$

Since the PF is 0.8 leading, the impedance phase angle is -36.9°; thus $\mathbf{Z}_p = 60\underline{/-36.9^\circ}\ \Omega$.

PRACTICE

12.5 A balanced three-phase three-wire system has a line voltage of 500 V. Two balanced Y-connected loads are present. One is a capacitive load with $7 - j2\ \Omega$ per phase, and the other is an inductive load with $4 + j2\ \Omega$ per phase. Find (*a*) the phase voltage; (*b*) the line current; (*c*) the total power drawn by the load; (*d*) the power factor at which the source is operating.

Ans: 289 V; 97.5 A; 83.0 kW; 0.983 lagging.

EXAMPLE 12.4

A balanced 600 W lighting load is added (in parallel) to the system of Example 12.3. Determine the new line current.

We first sketch a suitable per-phase circuit, as shown in Fig. 12.17. The 600 W load is assumed to be a balanced load evenly distributed among the three phases, resulting in an additional 200 W consumed by each phase.

The amplitude of the lighting current (labeled $\mathbf{I}_1$) is determined by

$$200 = \frac{300}{\sqrt{3}} |\mathbf{I}_1| \cos 0^\circ$$

so that

$$|\mathbf{I}_1| = 1.155 \text{ A}$$

In a similar way, the amplitude of the capacitive load current (labeled $\mathbf{I}_2$) is found to be unchanged from its previous value, since the voltage across it has remained the same:

$$|\mathbf{I}_2| = 2.89 \text{ A}$$

If we assume that the phase with which we are working has a phase voltage with an angle of 0°, then since $\cos^{-1}(0.8) = 36.9°$,

$$\mathbf{I}_1 = 1.155\underline{/0^\circ} \text{ A} \qquad \mathbf{I}_2 = 2.89\underline{/+36.9^\circ} \text{ A}$$

and the line current is

$$\mathbf{I}_L = \mathbf{I}_1 + \mathbf{I}_2 = 3.87\underline{/+26.6^\circ} \text{ A}$$

We can check our results by computing the power generated by this phase of the source

$$P_p = \frac{300}{\sqrt{3}} 3.87 \cos(+26.6^\circ) = 600 \text{ W}$$

which agrees with the fact that the individual phase is known to be supplying 200 W to the new lighting load, as well as 400 W to the original load.

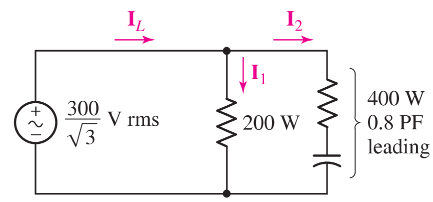


FIGURE 12.17 The per-phase circuit that is used to analyze a *balanced* three-phase example.

PRACTICE

12.6 Three balanced Y-connected loads are installed on a balanced three-phase four-wire system. Load 1 draws a total power of 6 kW at unity PF, load 2 pulls 10 kVA at PF = 0.96 lagging, and load 3 demands 7 kW at 0.85 lagging. If the phase voltage at the loads is 135 V, if each line has a resistance of 0.1 Ω, and if the neutral has a resistance of 1 Ω, find (*a*) the total power drawn by the loads; (*b*) the combined PF of the loads; (*c*) the total power lost in the four lines; (*d*) the phase voltage at the source; (*e*) the power factor at which the source is operating.

Ans: 22.6 kW; 0.954 lag; 1027 W; 140.6 V; 0.957 lagging.

If an *unbalanced* Y-connected load is present in an otherwise balanced three-phase system, the circuit may still be analyzed on a per-phase basis *if* the neutral wire is present and *if* it has zero impedance. If either of these conditions is not met, other methods must be used, such as mesh or nodal analysis. However, engineers who spend most of their time with unbalanced three-phase systems will find the use of *symmetrical components* a great time saver.

We leave this topic for more advanced texts.

12.4 THE DELTA (Δ) CONNECTION

An alternative to the Y-connected load is the Δ-connected configuration, as shown in Fig. 12.18. This type of configuration is very common, and does not possess a neutral connection.

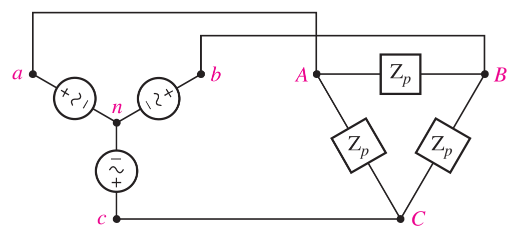


FIGURE 12.18 A balanced Δ-connected load is present on a three-wire three-phase system. The source happens to be Y-connected.

Let us consider a balanced Δ-connected load which consists of an impedance $\mathbf{Z}_p$ inserted between each pair of lines. With reference to Fig. 12.18, let us assume known line voltages

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}|$$

or known phase voltages

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

where

$$V_L = \sqrt{3}V_p \qquad \text{and} \qquad \mathbf{V}_{ab} = \sqrt{3}V_p\underline{/30^\circ}$$

as we found previously. Because the voltage across each branch of the Δ is known, the *phase currents* are easily found:

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_p} \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_p} \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_p}$$

and their differences provide us with the line currents, such as

$$\mathbf{I}_{aA} = \mathbf{I}_{AB} - \mathbf{I}_{CA}$$

Since we are working with a balanced system, the three phase currents are of equal amplitude:

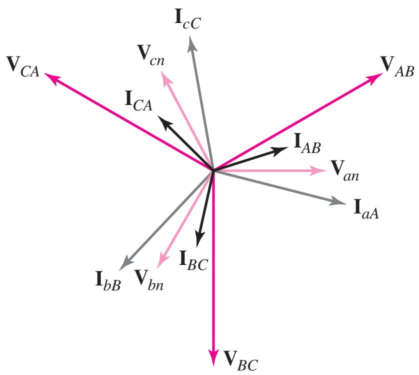
$$I_p = |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}|$$

The line currents are also equal in amplitude; the symmetry is apparent from the phasor diagram of Fig. 12.19. We thus have

$$I_L = |\mathbf{I}_{aA}| = |\mathbf{I}_{bB}| = |\mathbf{I}_{cC}|$$

and

$$I_L = \sqrt{3}I_p$$



■ **FIGURE 12.19** A phasor diagram that could apply to the circuit of Fig. 12.18 if $\mathbf{Z}_p$ were an inductive impedance.

Let us disregard the source for the moment and consider only the balanced load. If the load is Δ-connected, then the phase voltage and the line voltage are indistinguishable, but the line current is larger than the phase current by a factor of $\sqrt{3}$; with a Y-connected load, however, the phase current and the line current refer to the same current, and the line voltage is greater than the phase voltage by a factor of $\sqrt{3}$.

EXAMPLE 12.5

Determine the amplitude of the line current in a three-phase system with a line voltage of 300 V that supplies 1200 W to a Δ-connected load at a lagging PF of 0.8; then find the phase impedance.

Let us again consider a single phase. It draws 400 W, 0.8 lagging PF, at a 300 V line voltage. Thus,

$$400 = 300(I_p)(0.8)$$

and

$$I_p = 1.667 \text{ A}$$

and the relationship between phase currents and line currents yields

$$I_L = \sqrt{3}(1.667) = 2.89 \text{ A}$$

Next, the phase angle of the load is $\cos^{-1}(0.8) = 36.9°$, and therefore the impedance in each phase must be

$$\mathbf{Z}_p = \frac{300}{1.667}\underline{/36.9°} = 180\underline{/36.9°}\ \Omega$$

Again, keep in mind that we are assuming all voltages and currents are quoted as rms values.

PRACTICE

12.7 Each phase of a balanced three-phase Δ-connected load consists of a 200 mH inductor in series with the parallel combination of a 5 μF capacitor and a 200 Ω resistance. Assume zero line resistance and a phase voltage of 200 V at $\omega = 400$ rad/s. Find (a) the phase current; (b) the line current; (c) the total power absorbed by the load.

Ans: 1.158 A; 2.01 A; 693 W.

EXAMPLE 12.6

Determine the amplitude of the line current in a three-phase system with a 300 V line voltage that supplies 1200 W to a Y-connected load at a lagging PF of 0.8. (*This is the same circuit as in Example 12.5, but with a Y-connected load instead.*)

On a per-phase basis, we now have a phase voltage of $300/\sqrt{3}$ V, a power of 400 W, and a lagging PF of 0.8. Thus,

$$400 = \frac{300}{\sqrt{3}}(I_p)(0.8)$$

and

$$I_p = 2.89 \qquad (\text{and so } I_L = 2.89 \text{ A})$$

The phase angle of the load is again 36.9°, and thus the impedance in each phase of the Y is

$$\mathbf{Z}_p = \frac{300/\sqrt{3}}{2.89}\underline{/36.9^\circ} = 60\underline{/36.9^\circ}\ \Omega$$

The $\sqrt{3}$ factor not only relates phase and line quantities but also appears in a useful expression for the total power drawn by any balanced three-phase load. If we assume a Y-connected load with a power-factor angle θ, the power taken by any phase is

$$P_p = V_p I_p \cos\theta = V_p I_L \cos\theta = \frac{V_L}{\sqrt{3}} I_L \cos\theta$$

and the total power is

$$P = 3P_p = \sqrt{3} V_L I_L \cos\theta$$

In a similar way, the power delivered to each phase of a Δ-connected load is

$$P_p = V_p I_p \cos\theta = V_L I_p \cos\theta = V_L \frac{I_L}{\sqrt{3}} \cos\theta$$

giving a total power

$$P = 3P_p = \sqrt{3} V_L I_L \cos\theta \qquad [4]$$

Thus Eq. [4] enables us to calculate the total power delivered to a balanced load from a knowledge of the magnitude of the line voltage, of the line current, and of the phase angle of the load impedance (or admittance), regardless of whether the load is Y-connected or Δ-connected. The line current in

PRACTICE

12.8 A balanced three-phase three-wire system is terminated with two Δ-connected loads in parallel. Load 1 draws 40 kVA at a lagging PF of 0.8, while load 2 absorbs 24 kW at a leading PF of 0.9. Assume no line resistance, and let $\mathbf{V}_{ab} = 440\underline{/30^\circ}$ V. Find (*a*) the total power drawn by the loads; (*b*) the phase current $\mathbf{I}_{AB1}$ for the lagging load; (*c*) $\mathbf{I}_{AB2}$; (*d*) $\mathbf{I}_{aA}$.

Ans: 56.0 kW; $30.3\underline{/-6.87^\circ}$ A; $20.2\underline{/55.8^\circ}$ A; $75.3\underline{/-12.46^\circ}$ A.

Examples 12.5 and 12.6 can now be obtained in two simple steps:

$$1200 = \sqrt{3}(300)(I_L)(0.8)$$

Therefore,

$$I_L = \frac{5}{\sqrt{3}} = 2.89 \text{ A}$$

A brief comparison of phase and line voltages as well as phase and line currents is presented in Table 12.1 for both Y- and Δ-connected loads powered by a Y-connected three-phase source.

TABLE 12.1 Comparison of Y- and Δ-Connected Three-Phase Loads. V_p Is the Voltage Magnitude of Each Y-Connected *Source* Phase

Load	Phase Voltage	Line Voltage	Phase Current	Line Current	Power per Phase
Y	$\mathbf{V}_{AN} = V_p\underline{/0^\circ}$ $\mathbf{V}_{BN} = V_p\underline{/-120^\circ}$ $\mathbf{V}_{CN} = V_p\underline{/-240^\circ}$	$\mathbf{V}_{AB} = \mathbf{V}_{ab}$ $= (\sqrt{3}\underline{/30^\circ})\mathbf{V}_{AN}$ $= \sqrt{3}V_p\underline{/30^\circ}$ $\mathbf{V}_{BC} = \mathbf{V}_{bc}$ $= (\sqrt{3}\underline{/30^\circ})\mathbf{V}_{BN}$ $= \sqrt{3}V_p\underline{/-90^\circ}$ $\mathbf{V}_{CA} = \mathbf{V}_{ca}$ $= (\sqrt{3}\underline{/30^\circ})\mathbf{V}_{CN}$ $= \sqrt{3}V_p\underline{/-210^\circ}$	$\mathbf{I}_{aA} = \mathbf{I}_{AN} = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_p}$ $\mathbf{I}_{bB} = \mathbf{I}_{BN} = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_p}$ $\mathbf{I}_{cC} = \mathbf{I}_{CN} = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_p}$	$\mathbf{I}_{aA} = \mathbf{I}_{AN} = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_p}$ $\mathbf{I}_{bB} = \mathbf{I}_{BN} = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_p}$ $\mathbf{I}_{cC} = \mathbf{I}_{CN} = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_p}$	$\sqrt{3}V_L I_L \cos\theta$ where $\cos\theta =$ power factor of the load
Δ	$\mathbf{V}_{AB} = \mathbf{V}_{ab}$ $= \sqrt{3}V_p\underline{/30^\circ}$ $\mathbf{V}_{BC} = \mathbf{V}_{bc}$ $= \sqrt{3}V_p\underline{/-90^\circ}$ $\mathbf{V}_{CA} = \mathbf{V}_{ca}$ $= \sqrt{3}V_p\underline{/-210^\circ}$	$\mathbf{V}_{AB} = \mathbf{V}_{ab}$ $= \sqrt{3}V_p\underline{/30^\circ}$ $\mathbf{V}_{BC} = \mathbf{V}_{bc}$ $= \sqrt{3}V_p\underline{/-90^\circ}$ $\mathbf{V}_{CA} = \mathbf{V}_{ca}$ $= \sqrt{3}V_p\underline{/-210^\circ}$	$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_p}$ $\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_p}$ $\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_p}$	$\mathbf{I}_{aA} = (\sqrt{3}\underline{/-30^\circ})\frac{\mathbf{V}_{AB}}{\mathbf{Z}_p}$ $\mathbf{I}_{bB} = (\sqrt{3}\underline{/-30^\circ})\frac{\mathbf{V}_{BC}}{\mathbf{Z}_p}$ $\mathbf{I}_{cC} = (\sqrt{3}\underline{/-30^\circ})\frac{\mathbf{V}_{CA}}{\mathbf{Z}_p}$	$\sqrt{3}V_L I_L \cos\theta$ where $\cos\theta =$ power factor of the load

Δ-Connected Sources

The source may also be connected in a Δ configuration. This is not typical, however, for a slight unbalance in the source phases can lead to large currents circulating in the Δ loop. For example, let us call the three single-phase sources $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{cd}$. Before closing the Δ by connecting d to a, let us determine the unbalance by measuring the sum $\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca}$. Suppose that the amplitude of the result is only 1 percent of the line voltage. The circulating current is thus approximately $\frac{1}{3}$ percent of the line voltage divided by the internal impedance of any source. How large is this impedance apt to be? It must depend on the current that the source is expected to deliver with a negligible drop in terminal voltage. If we assume that this maximum current causes a 1 percent drop in the terminal voltage, then *the circulating current is one-third of the maximum current*! This reduces the useful current capacity of the source and also increases the losses in the system.

chapter

12

Three-Phase Circuits

He who cannot forgive others breaks the bridge over which he must pass himself.

—G. Herbert

Enhancing Your Skills and Your Career

ABET EC 2000 criteria (3.e), *"an ability to identify, formulate, and solve engineering problems."*

Developing and enhancing your "ability to identify, formulate, and solve engineering problems" is a primary focus of textbook. Following our six step problem-solving process is the best way to practice this skill. Our recommendation is that you use this process whenever possible. You may be pleased to learn that this process works well for nonengineering courses.

ABET EC 2000 criteria (f), *"an understanding of professional and ethical responsibility."*

"An understanding of professional and ethical responsibility" is required of every engineer. To some extent, this understanding is very personal for each of us. Let us identify some pointers to help you develop this understanding. One of my favorite examples is that an engineer has the responsibility to answer what I call the "unasked question." For instance, assume that you own a car that has a problem with the transmission. In the process of selling that car, the prospective buyer asks you if there is a problem in the right-front wheel bearing. You answer no. However, as an engineer, you are required to inform the buyer that there is a problem with the transmission without being asked.

Your responsibility both professionally and ethically is to perform in a manner that does not harm those around you and to whom you are responsible. Clearly, developing this capability will take time and maturity on your part. I recommend practicing this by looking for professional and ethical components in your day-to-day activities.

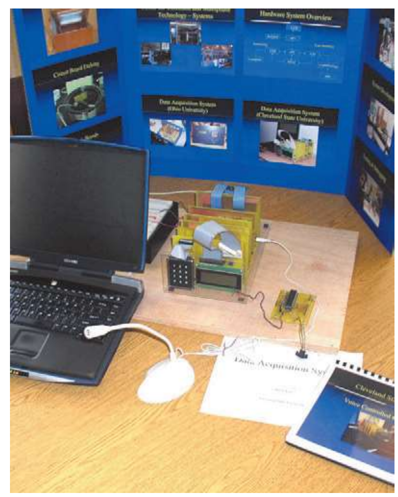


Photo by Charles Alexander

12.1 Introduction

So far in this text, we have dealt with single-phase circuits. A single-phase ac power system consists of a generator connected through a pair of wires (a transmission line) to a load. Figure 12.1(a) depicts a single-phase two-wire system, where V_p is the rms magnitude of the source voltage and ϕ is the phase. What is more common in practice is a single-phase three-wire system, shown in Fig. 12.1(b). It contains two identical sources (equal magnitude and the same phase) that are connected to two loads by two outer wires and the neutral. For example, the normal household system is a single-phase three-wire system because the terminal voltages have the same magnitude and the same phase. Such a system allows the connection of both 120-V and 240-V appliances.

Historical note: Thomas Edison invented a *three-wire system*, using three wires instead of four.

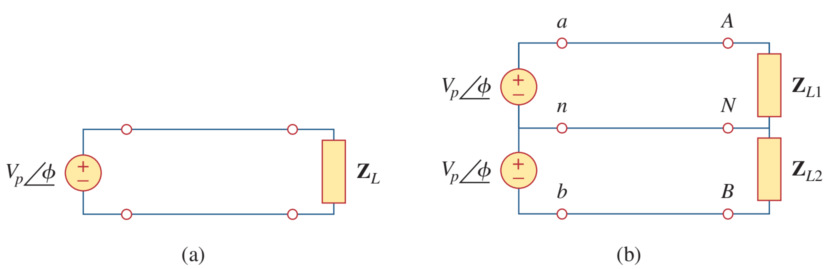


Figure 12.1
Single-phase systems: (a) two-wire type, (b) three-wire type.

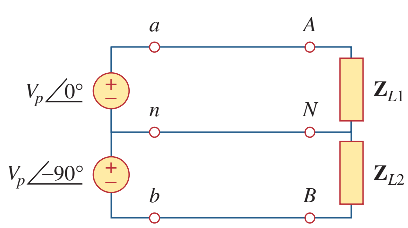


Figure 12.2
Two-phase three-wire system.

Circuits or systems in which the ac sources operate at the same frequency but different phases are known as *polyphase*. Figure 12.2 shows a two-phase three-wire system, and Fig. 12.3 shows a three-phase four-wire system. As distinct from a single-phase system, a two-phase system is produced by a generator consisting of two coils placed perpendicular to each other so that the voltage generated by one lags the other by 90°. By the same token, a three-phase system is produced by a generator consisting of three sources having the same amplitude and frequency but out of phase with each other by 120°. Since the three-phase system is by far the most prevalent and most economical polyphase system, discussion in this chapter is mainly on three-phase systems.

Three-phase systems are important for at least three reasons. First, nearly all electric power is generated and distributed in three-phase, at the operating frequency of 60 Hz (or $\omega = 377$ rad/s) in the United States or 50 Hz (or $\omega = 314$ rad/s) in some other parts of the world. When one-phase or two-phase inputs are required, they are taken from the three-phase system rather than generated independently. Even when more than three phases are needed—such as in the aluminum industry, where 48 phases are required for melting purposes—they can be provided by manipulating the three phases supplied. Second, the instantaneous power in a three-phase system can be constant (not pulsating), as we will see in Section 12.7. This results in uniform power transmission and less vibration of three-phase machines. Third, for the same amount of power, the three-phase system is more economical than the single-phase. The amount of wire required for a three-phase system is less than that required for an equivalent single-phase system.

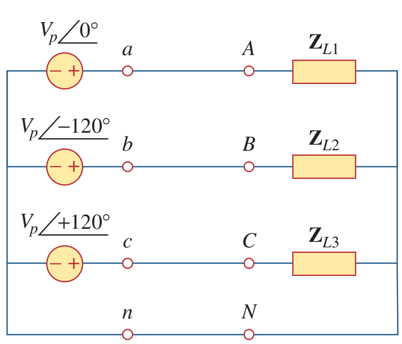


Figure 12.3
Three-phase four-wire system.

Historical

Nikola Tesla (1856–1943) was a Croatian-American engineer whose inventions—among them the induction motor and the first polyphase ac power system—greatly influenced the settlement of the ac versus dc debate in favor of ac. He was also responsible for the adoption of 60 Hz as the standard for ac power systems in the United States.

Born in Austria-Hungary (now Croatia), to a clergyman, Tesla had an incredible memory and a keen affinity for mathematics. He moved to the United States in 1884 and first worked for Thomas Edison. At that time, the country was in the "battle of the currents" with George Westinghouse (1846–1914) promoting ac and Thomas Edison rigidly leading the dc forces. Tesla left Edison and joined Westinghouse because of his interest in ac. Through Westinghouse, Tesla gained the reputation and acceptance of his polyphase ac generation, transmission, and distribution system. He held 700 patents in his lifetime. His other inventions include high-voltage apparatus (the tesla coil) and a wireless transmission system. The unit of magnetic flux density, the tesla, was named in honor of him.

Courtesy Smithsonian Institution

We begin with a discussion of balanced three-phase voltages. Then we analyze each of the four possible configurations of balanced three-phase systems. We also discuss the analysis of unbalanced three-phase systems. We learn how to use *PSpice for Windows* to analyze a balanced or unbalanced three-phase system. Finally, we apply the concepts developed in this chapter to three-phase power measurement and residential electrical wiring.

12.2 Balanced Three-Phase Voltages

Three-phase voltages are often produced with a three-phase ac generator (or alternator) whose cross-sectional view is shown in Fig. 12.4. The generator basically consists of a rotating magnet (called the *rotor*) surrounded by a stationary winding (called the *stator*). Three separate

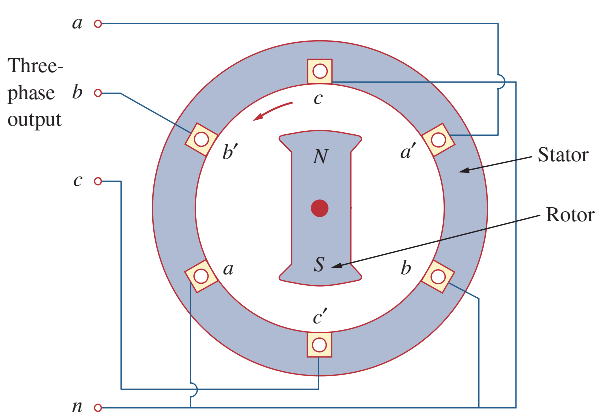


Figure 12.4
A three-phase generator.

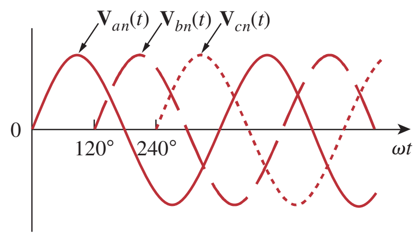


Figure 12.5
The generated voltages are 120° apart from each other.

windings or coils with terminals a-a', b-b', and c-c' are physically placed 120° apart around the stator. Terminals a and a', for example, stand for one of the ends of coils going into and the other end coming out of the page. As the rotor rotates, its magnetic field "cuts" the flux from the three coils and induces voltages in the coils. Because the coils are placed 120° apart, the induced voltages in the coils are equal in magnitude but out of phase by 120° (Fig. 12.5). Since each coil can be regarded as a single-phase generator by itself, the three-phase generator can supply power to both single-phase and three-phase loads.

A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines). (Three-phase current sources are very scarce.) A three-phase system is equivalent to three single-phase circuits. The voltage sources can be either wye-connected as shown in Fig. 12.6(a) or delta-connected as in Fig. 12.6(b).

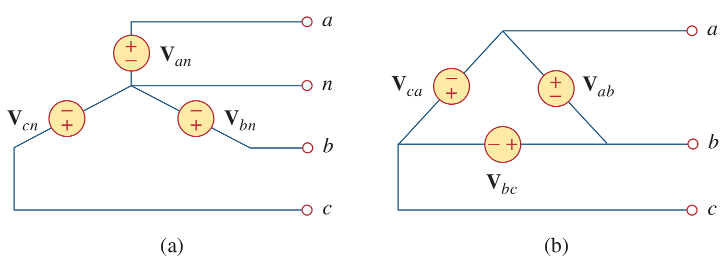


Figure 12.6
Three-phase voltage sources: (a) Y-connected source, (b) Δ-connected source.

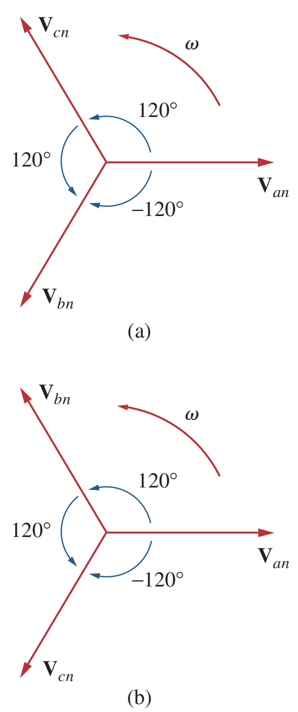


Figure 12.7
Phase sequences: (a) *abc* or positive sequence, (b) *acb* or negative sequence.

Let us consider the wye-connected voltages in Fig. 12.6(a) for now. The voltages $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are respectively between lines a, b, and c, and the neutral line n. These voltages are called *phase voltages*. If the voltage sources have the same amplitude and frequency ω and are out of phase with each other by 120°, the voltages are said to be *balanced*. This implies that

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0 \tag{12.1}$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.2}$$

Thus,

> **Balanced phase voltages** are equal in magnitude and are out of phase with each other by 120°.

Since the three-phase voltages are 120° out of phase with each other, there are two possible combinations. One possibility is shown in Fig. 12.7(a) and expressed mathematically as

$$\begin{aligned} \mathbf{V}_{an} &= V_p\underline{/0^\circ} \\ \mathbf{V}_{bn} &= V_p\underline{/-120^\circ} \\ \mathbf{V}_{cn} &= V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ} \end{aligned} \tag{12.3}$$

where V_p is the effective or rms value of the phase voltages. This is known as the *abc sequence* or *positive sequence*. In this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{bn}$, which in turn leads $\mathbf{V}_{cn}$. This sequence is produced when the rotor in Fig. 12.4 rotates counterclockwise. The other possibility is shown in Fig. 12.7(b) and is given by

As a common tradition in power systems, voltage and current in this chapter are in rms values unless otherwise stated.

$$\begin{aligned} \mathbf{V}_{an} &= V_p\underline{/0^\circ} \\ \mathbf{V}_{cn} &= V_p\underline{/-120^\circ} \\ \mathbf{V}_{bn} &= V_p\underline{/-240^\circ} = V_p\underline{/+120^\circ} \end{aligned} \tag{12.4}$$

This is called the *acb sequence* or *negative sequence*. For this phase sequence, $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$, which in turn leads $\mathbf{V}_{bn}$. The *acb* sequence is produced when the rotor in Fig. 12.4 rotates in the clockwise direction. It is easy to show that the voltages in Eqs. (12.3) or (12.4) satisfy Eqs. (12.1) and (12.2). For example, from Eq. (12.3),

$$\begin{aligned} \mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} &= V_p\underline{/0^\circ} + V_p\underline{/-120^\circ} + V_p\underline{/+120^\circ} \\ &= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866) \\ &= 0 \end{aligned} \tag{12.5}$$

The **phase sequence** is the time order in which the voltages pass through their respective maximum values.

The phase sequence may also be regarded as the order in which the phase voltages reach their peak (or maximum) values with respect to time.

The phase sequence is determined by the order in which the phasors pass through a fixed point in the phase diagram.

Reminder: As time increases, each phasor (or sinor) rotates at an angular velocity ω.

In Fig. 12.7(a), as the phasors rotate in the counterclockwise direction with frequency ω, they pass through the horizontal axis in a sequence *abcabca* . . . . Thus, the sequence is *abc* or *bca* or *cab*. Similarly, for the phasors in Fig. 12.7(b), as they rotate in the counterclockwise direction, they pass the horizontal axis in a sequence *acbacba* . . . . This describes the *acb* sequence. The phase sequence is important in three-phase power distribution. It determines the direction of the rotation of a motor connected to the power source, for example.

Like the generator connections, a three-phase load can be either wye-connected or delta-connected, depending on the end application. Figure 12.8(a) shows a wye-connected load, and Fig. 12.8(b) shows a delta-connected load. The neutral line in Fig. 12.8(a) may or may not be there, depending on whether the system is four- or three-wire. (And, of course, a neutral connection is topologically impossible for a delta connection.) A wye- or delta-connected load is said to be *unbalanced* if the phase impedances are not equal in magnitude or phase.

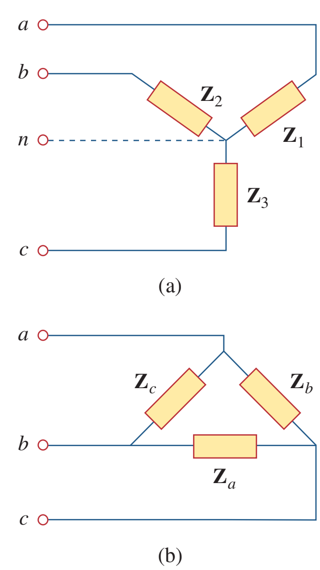


Figure 12.8
Two possible three-phase load configurations: (a) a Y-connected load, (b) a Δ-connected load.

A **balanced load** is one in which the phase impedances are equal in magnitude and in phase.

For a *balanced* wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y \tag{12.6}$$

Reminder: A Y-connected load consists of three impedances connected to a neutral node, while a Δ-connected load consists of three impedances connected around a loop. The load is balanced when the three impedances are equal in either case.

where $\mathbf{Z}_Y$ is the load impedance per phase. For a *balanced* delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta \tag{12.7}$$

where $\mathbf{Z}_\Delta$ is the load impedance per phase in this case. We recall from Eq. (9.69) that

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta \tag{12.8}$$

so we know that a wye-connected load can be transformed into a delta-connected load, or vice versa, using Eq. (12.8).

Since both the three-phase source and the three-phase load can be either wye- or delta-connected, we have four possible connections:

- Y-Y connection (i.e., Y-connected source with a Y-connected load).
- Y-Δ connection.
- Δ-Δ connection.
- Δ-Y connection.

In subsequent sections, we will consider each of these possible configurations.

It is appropriate to mention here that a balanced delta-connected load is more common than a balanced wye-connected load. This is due to the ease with which loads may be added or removed from each phase of a delta-connected load. This is very difficult with a wye-connected load because the neutral may not be accessible. On the other hand, delta-connected sources are not common in practice because of the circulating current that will result in the delta-mesh if the three-phase voltages are slightly unbalanced.

Example 12.1

Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$
$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

Solution:

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\underline{/10^\circ}\text{ V}, \qquad \mathbf{V}_{bn} = 200\underline{/-230^\circ}\text{ V}, \qquad \mathbf{V}_{cn} = 200\underline{/-110^\circ}\text{ V}$$

We notice that $\mathbf{V}_{an}$ leads $\mathbf{V}_{cn}$ by 120° and $\mathbf{V}_{cn}$ in turn leads $\mathbf{V}_{bn}$ by 120°. Hence, we have an *acb* sequence.

Practice Problem 12.1

Given that $\mathbf{V}_{bn} = 110\underline{/30^\circ}$ V, find $\mathbf{V}_{an}$ and $\mathbf{V}_{cn}$, assuming a positive (*abc*) sequence.

Answer: $110\underline{/150^\circ}$ V, $110\underline{/-90^\circ}$ V.

12.3 Balanced Wye-Wye Connection

We begin with the Y-Y system, because any balanced three-phase system can be reduced to an equivalent Y-Y system. Therefore, analysis of this system should be regarded as the key to solving all balanced three-phase systems.

> A **balanced Y-Y system** is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.

Consider the balanced four-wire Y-Y system of Fig. 12.9, where a Y-connected load is connected to a Y-connected source. We assume a balanced load so that load impedances are equal. Although the impedance $\mathbf{Z}_Y$ is the total load impedance per phase, it may also be regarded as the sum of the source impedance $\mathbf{Z}_s$, line impedance $\mathbf{Z}_\ell$, and load impedance $\mathbf{Z}_L$ for each phase, since these impedances are in series. As illustrated in Fig. 12.9, $\mathbf{Z}_s$ denotes the internal impedance of the phase winding of the generator; $\mathbf{Z}_\ell$ is the impedance of the line joining a phase of the source with a phase of the load; $\mathbf{Z}_L$ is the impedance of each phase of the load; and $\mathbf{Z}_n$ is the impedance of the neutral line. Thus, in general

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L \tag{12.9}$$

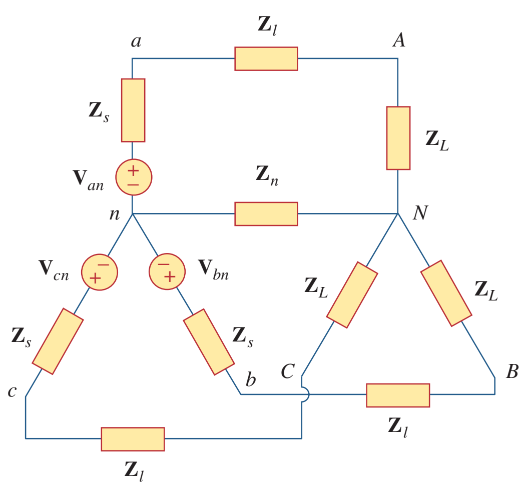


Figure 12.9
A balanced Y-Y system, showing the source, line, and load impedances.

$\mathbf{Z}_s$ and $\mathbf{Z}_\ell$ are often very small compared with $\mathbf{Z}_L$, so one can assume that $\mathbf{Z}_Y = \mathbf{Z}_L$ if no source or line impedance is given. In any event, by lumping the impedances together, the Y-Y system in Fig. 12.9 can be simplified to that shown in Fig. 12.10.

Assuming the positive sequence, the *phase* voltages (or line-to-neutral voltages) are

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ \tag{12.10}$$

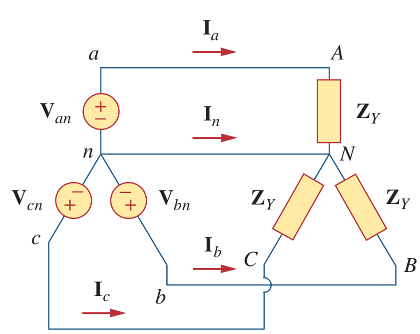


Figure 12.10
Balanced Y-Y connection.

The *line-to-line* voltages or simply *line* voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are related to the phase voltages. For example,

$$\begin{aligned}\mathbf{V}_{ab} &= \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p\underline{/0^\circ} - V_p\underline{/-120^\circ} \\ &= V_p\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \sqrt{3}V_p\underline{/30^\circ}\end{aligned} \tag{12.11a}$$

Similarly, we can obtain

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p\underline{/-90^\circ} \tag{12.11b}$$

$$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p\underline{/-210^\circ} \tag{12.11c}$$

Thus, the magnitude of the line voltages V_L is $\sqrt{3}$ times the magnitude of the phase voltages V_p, or

$$\boxed{V_L = \sqrt{3}V_p} \tag{12.12}$$

where

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}| \tag{12.13}$$

and

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}| \tag{12.14}$$

Also the line voltages lead their corresponding phase voltages by 30°. Figure 12.11(a) illustrates this. Figure 12.11(a) also shows how to determine $\mathbf{V}_{ab}$ from the phase voltages, while Fig. 12.11(b) shows the same for the three line voltages. Notice that $\mathbf{V}_{ab}$ leads $\mathbf{V}_{bc}$ by 120°, and $\mathbf{V}_{bc}$ leads $\mathbf{V}_{ca}$ by 120°, so that the line voltages sum up to zero as do the phase voltages.

Applying KVL to each phase in Fig. 12.10, we obtain the line currents as

$$\begin{aligned}\mathbf{I}_a &= \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\underline{/-120^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\underline{/-120^\circ} \\ \mathbf{I}_c &= \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\underline{/-240^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\underline{/-240^\circ}\end{aligned} \tag{12.15}$$

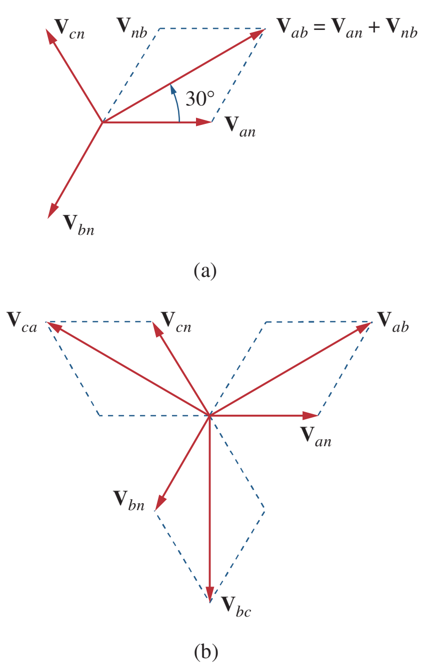


Figure 12.11
Phasor diagrams illustrating the relationship between line voltages and phase voltages.

We can readily infer that the line currents add up to zero,

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0 \tag{12.16}$$

so that

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0 \tag{12.17a}$$

or

$$\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0 \tag{12.17b}$$

that is, the voltage across the neutral wire is zero. The neutral line can thus be removed without affecting the system. In fact, in long distance power transmission, conductors in multiples of three are used with the earth itself acting as the neutral conductor. Power systems designed in this way are well grounded at all critical points to ensure safety.

While the *line* current is the current in each line, the *phase* current is the current in each phase of the source or load. In the Y-Y system, the line current is the same as the phase current. We will use single subscripts for line currents because it is natural and conventional to assume that line currents flow from the source to the load.

An alternative way of analyzing a balanced Y-Y system is to do so on a "per phase" basis. We look at one phase, say phase a, and analyze the single-phase equivalent circuit in Fig. 12.12. The single-phase analysis yields the line current $\mathbf{I}_a$ as

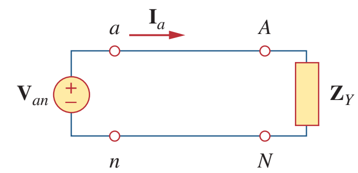


Figure 12.12
A single-phase equivalent circuit.

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} \tag{12.18}$$

From $\mathbf{I}_a$, we use the phase sequence to obtain other line currents. Thus, as long as the system is balanced, we need only analyze one phase. We may do this even if the neutral line is absent, as in the three-wire system.

Example 12.2

Calculate the line currents in the three-wire Y-Y system of Fig. 12.13.

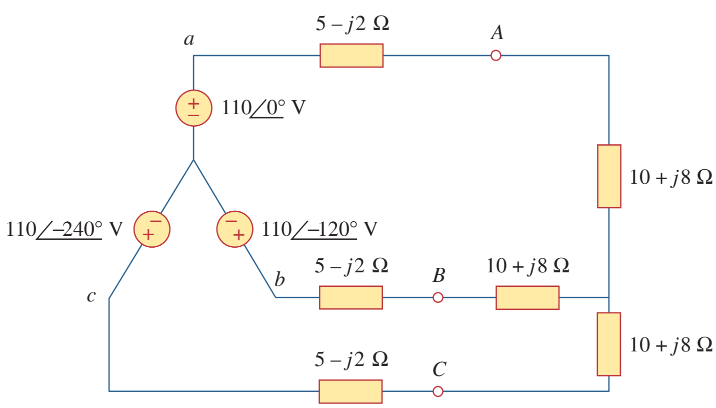


Figure 12.13
Three-wire Y-Y system; for Example 12.2.

Solution:
The three-phase circuit in Fig. 12.13 is balanced; we may replace it with its single-phase equivalent circuit such as in Fig. 12.12. We obtain $\mathbf{I}_a$ from the single-phase analysis as

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}$$

where $\mathbf{Z}_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\underline{/21.8^\circ}$. Hence,

$$\mathbf{I}_a = \frac{110\underline{/0^\circ}}{16.155\underline{/21.8^\circ}} = 6.81\underline{/-21.8^\circ}\text{ A}$$

Since the source voltages in Fig. 12.13 are in positive sequence, the line currents are also in positive sequence:

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 6.81\underline{/-141.8^\circ}\text{ A}$$

$$\mathbf{I}_c = \mathbf{I}_a\underline{/-240^\circ} = 6.81\underline{/-261.8^\circ}\text{ A} = 6.81\underline{/98.2^\circ}\text{ A}$$

Practice Problem 12.2

A Y-connected balanced three-phase generator with an impedance of $0.4 + j0.3\ \Omega$ per phase is connected to a Y-connected balanced load with an impedance of $24 + j19\ \Omega$ per phase. The line joining the generator and the load has an impedance of $0.6 + j0.7\ \Omega$ per phase. Assuming a positive sequence for the source voltages and that $\mathbf{V}_{an} = 120\underline{/30^\circ}$ V, find: (a) the line voltages, (b) the line currents.

Answer: (a) $207.8\underline{/60^\circ}$ V, $207.8\underline{/-60^\circ}$ V, $207.8\underline{/-180^\circ}$ V, (b) $3.75\underline{/-8.66^\circ}$ A, $3.75\underline{/-128.66^\circ}$ A, $3.75\underline{/111.34^\circ}$ A.

12.4 Balanced Wye-Delta Connection

A **balanced Y-Δ system** consists of a balanced Y-connected source feeding a balanced Δ-connected load.

This is perhaps the most practical three-phase system, as the three-phase sources are usually Y-connected while the three-phase loads are usually Δ-connected.

The balanced Y-delta system is shown in Fig. 12.14, where the source is Y-connected and the load is Δ-connected. There is, of course, no neutral connection from source to load for this case. Assuming the positive sequence, the phase voltages are again

$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$
$$\mathbf{V}_{bn} = V_p\underline{/-120^\circ}, \qquad \mathbf{V}_{cn} = V_p\underline{/+120^\circ} \tag{12.19}$$

As shown in Section 12.3, the line voltages are

$$\mathbf{V}_{ab} = \sqrt{3}V_p\underline{/30^\circ} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \sqrt{3}V_p\underline{/-90^\circ} = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \sqrt{3}V_p\underline{/-150^\circ} = \mathbf{V}_{CA} \tag{12.20}$$

showing that the line voltages are equal to the voltages across the load impedances for this system configuration. From these voltages, we can obtain the phase currents as

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} \tag{12.21}$$

These currents have the same magnitude but are out of phase with each other by 120°.

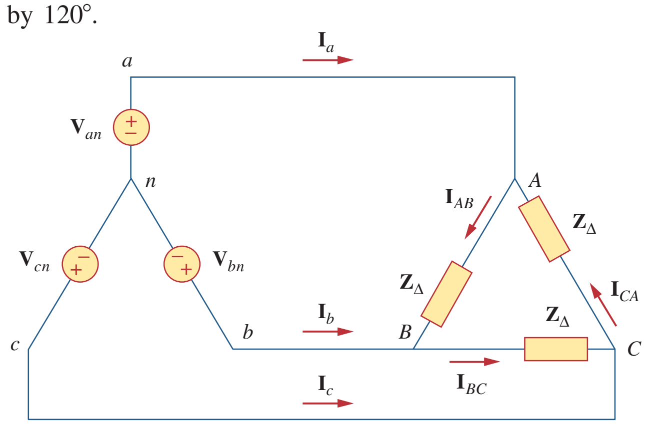


Figure 12.14
Balanced Y-Δ connection.

Another way to get these phase currents is to apply KVL. For example, applying KVL around loop *aABbna* gives

$$-\mathbf{V}_{an} + \mathbf{Z}_\Delta \mathbf{I}_{AB} + \mathbf{V}_{bn} = 0$$

or

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{an} - \mathbf{V}_{bn}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} \tag{12.22}$$

which is the same as Eq. (12.21). This is the more general way of finding the phase currents.

The line currents are obtained from the phase currents by applying KCL at nodes *A*, *B*, and *C*. Thus,

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.23}$$

Since $\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle{-240^\circ}$,

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA} &= \mathbf{I}_{AB}(1 - 1\angle{-240^\circ}) \\ &= \mathbf{I}_{AB}(1 + 0.5 - j0.866) = \mathbf{I}_{AB}\sqrt{3}\angle{-30^\circ}\end{aligned} \tag{12.24}$$

showing that the magnitude I_L of the line current is $\sqrt{3}$ times the magnitude I_p of the phase current, or

$$\boxed{I_L = \sqrt{3} I_p} \tag{12.25}$$

where

$$I_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c| \tag{12.26}$$

and

$$I_p = |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| \tag{12.27}$$

Also, the line currents lag the corresponding phase currents by 30°, assuming the positive sequence. Figure 12.15 is a phasor diagram illustrating the relationship between the phase and line currents.

An alternative way of analyzing the Y-Δ circuit is to transform the Δ-connected load to an equivalent Y-connected load. Using the Δ-Y transformation formula in Eq. (12.8),

$$\boxed{\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3}} \tag{12.28}$$

After this transformation, we now have a Y-Y system as in Fig. 12.10. The three-phase Y-Δ system in Fig. 12.14 can be replaced by the single-phase equivalent circuit in Fig. 12.16. This allows us to calculate only the line currents. The phase currents are obtained using Eq. (12.25) and utilizing the fact that each of the phase currents leads the corresponding line current by 30°.

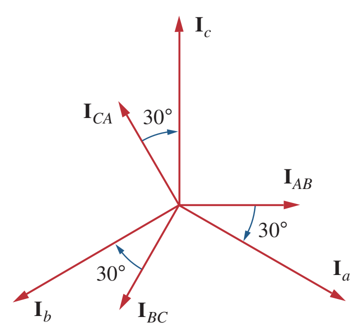


Figure 12.15
Phasor diagram illustrating the relationship between phase and line currents.

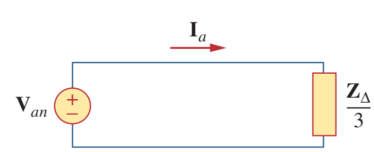


Figure 12.16
A single-phase equivalent circuit of a balanced Y-Δ circuit.

Example 12.3

A balanced *abc*-sequence Y-connected source with $\mathbf{V}_{an} = 100\angle{10^\circ}$ V is connected to a Δ-connected balanced load $(8 + j4)\ \Omega$ per phase. Calculate the phase and line currents.

Solution:
This can be solved in two ways.

■ **METHOD 1** The load impedance is

$$\mathbf{Z}_\Delta = 8 + j4 = 8.944\underline{/26.57^\circ}\ \Omega$$

If the phase voltage $\mathbf{V}_{an} = 100\underline{/10^\circ}$, then the line voltage is

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\underline{/30^\circ} = 100\sqrt{3}\underline{/10^\circ + 30^\circ} = \mathbf{V}_{AB}$$

or

$$\mathbf{V}_{AB} = 173.2\underline{/40^\circ}\text{ V}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\underline{/40^\circ}}{8.944\underline{/26.57^\circ}} = 19.36\underline{/13.43^\circ}\text{ A}$$
$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 19.36\underline{/-106.57^\circ}\text{ A}$$
$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 19.36\underline{/133.43^\circ}\text{ A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = \sqrt{3}(19.36)\underline{/13.43^\circ - 30^\circ}$$
$$= 33.53\underline{/-16.57^\circ}\text{ A}$$
$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 33.53\underline{/-136.57^\circ}\text{ A}$$
$$\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ} = 33.53\underline{/103.43^\circ}\text{ A}$$

■ **METHOD 2** Alternatively, using single-phase analysis,

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_\Delta/3} = \frac{100\underline{/10^\circ}}{2.981\underline{/26.57^\circ}} = 33.54\underline{/-16.57^\circ}\text{ A}$$

as above. Other line currents are obtained using the *abc* phase sequence.

Practice Problem 12.3

One line voltage of a balanced Y-connected source is $\mathbf{V}_{AB} = 120\underline{/-20^\circ}$ V. If the source is connected to a Δ-connected load of $20\underline{/40^\circ}\ \Omega$, find the phase and line currents. Assume the *abc* sequence.

Answer: $6\underline{/-60^\circ}$ A, $6\underline{/-180^\circ}$ A, $6\underline{/60^\circ}$ A, $10.392\underline{/-90^\circ}$ A, $10.392\underline{/150^\circ}$ A, $10.392\underline{/30^\circ}$ A.

12.5 Balanced Delta-Delta Connection

A **balanced Δ-Δ system** is one in which both the balanced source and balanced load are Δ-connected.

The source as well as the load may be delta-connected as shown in Fig. 12.17. Our goal is to obtain the phase and line currents as usual.

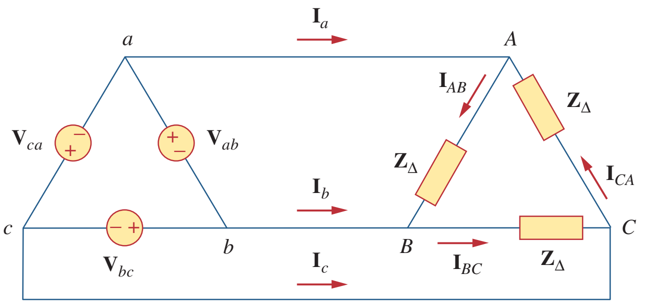


Figure 12.17
A balanced Δ-Δ connection.

Assuming a positive sequence, the phase voltages for a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bc} = V_p\angle -120^\circ, \qquad \mathbf{V}_{ca} = V_p\angle +120^\circ \tag{12.29}$$

The line voltages are the same as the phase voltages. From Fig. 12.17, assuming there is no line impedances, the phase voltages of the delta-connected source are equal to the voltages across the impedances; that is,

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA} \tag{12.30}$$

Hence, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{Z_\Delta} = \frac{\mathbf{V}_{ab}}{Z_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{Z_\Delta} = \frac{\mathbf{V}_{bc}}{Z_\Delta}$$
$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{Z_\Delta} = \frac{\mathbf{V}_{ca}}{Z_\Delta} \tag{12.31}$$

Since the load is delta-connected just as in the previous section, some of the formulas derived there apply here. The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C, as we did in the previous section:

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC} \tag{12.32}$$

Also, as shown in the last section, each line current lags the corresponding phase current by 30°; the magnitude I_L of the line current is $\sqrt{3}$ times the magnitude I_p of the phase current,

$$I_L = \sqrt{3} I_p \tag{12.33}$$

An alternative way of analyzing the Δ-Δ circuit is to convert both the source and the load to their Y equivalents. We already know that $\mathbf{Z}_Y = \mathbf{Z}_\Delta/3$. To convert a Δ-connected source to a Y-connected source, see the next section.

Example 12.4

A balanced Δ-connected load having an impedance $20 - j15\ \Omega$ is connected to a Δ-connected, positive-sequence generator having $\mathbf{V}_{ab} = 330\angle 0^\circ$ V. Calculate the phase currents of the load and the line currents.

Solution:
The load impedance per phase is

$$\mathbf{Z}_\Delta = 20 - j15 = 25\underline{/-36.87^\circ}\ \Omega$$

Since $\mathbf{V}_{AB} = \mathbf{V}_{ab}$, the phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\underline{/0^\circ}}{25\underline{/-36.87}} = 13.2\underline{/36.87^\circ}\text{ A}$$
$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 13.2\underline{/-83.13^\circ}\text{ A}$$
$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 13.2\underline{/156.87^\circ}\text{ A}$$

For a delta load, the line current always lags the corresponding phase current by 30° and has a magnitude $\sqrt{3}$ times that of the phase current. Hence, the line currents are

$$\begin{aligned}\mathbf{I}_a &= \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = (13.2\underline{/36.87^\circ})(\sqrt{3}\underline{/-30^\circ})\\ &= 22.86\underline{/6.87^\circ}\text{ A}\\ \mathbf{I}_b &= \mathbf{I}_a\underline{/-120^\circ} = 22.86\underline{/-113.13^\circ}\text{ A}\\ \mathbf{I}_c &= \mathbf{I}_a\underline{/+120^\circ} = 22.86\underline{/126.87^\circ}\text{ A}\end{aligned}$$

Practice Problem 12.4

A positive-sequence, balanced Δ-connected source supplies a balanced Δ-connected load. If the impedance per phase of the load is $18 + j12\ \Omega$ and $\mathbf{I}_a = 9.609\underline{/35^\circ}$ A, find $\mathbf{I}_{AB}$ and $\mathbf{V}_{AB}$.

Answer: $5.548\underline{/65^\circ}$ A, $120\underline{/98.69^\circ}$ V.

12.6 Balanced Delta-Wye Connection

A **balanced Δ-Y system** consists of a balanced Δ-connected source feeding a balanced Y-connected load.

Consider the Δ-Y circuit in Fig. 12.18. Again, assuming the *abc* sequence, the phase voltages of a delta-connected source are

$$\mathbf{V}_{ab} = V_p\underline{/0^\circ}, \qquad \mathbf{V}_{bc} = V_p\underline{/-120^\circ}$$
$$\mathbf{V}_{ca} = V_p\underline{/+120^\circ} \tag{12.34}$$

These are also the line voltages as well as the phase voltages.

We can obtain the line currents in many ways. One way is to apply KVL to loop *aANBba* in Fig. 12.18, writing

$$-\mathbf{V}_{ab} + \mathbf{Z}_Y\mathbf{I}_a - \mathbf{Z}_Y\mathbf{I}_b = 0$$

or

$$\mathbf{Z}_Y(\mathbf{I}_a - \mathbf{I}_b) = \mathbf{V}_{ab} = V_p\underline{/0^\circ}$$

Thus,

$$\mathbf{I}_a - \mathbf{I}_b = \frac{V_p\underline{/0^\circ}}{\mathbf{Z}_Y} \tag{12.35}$$

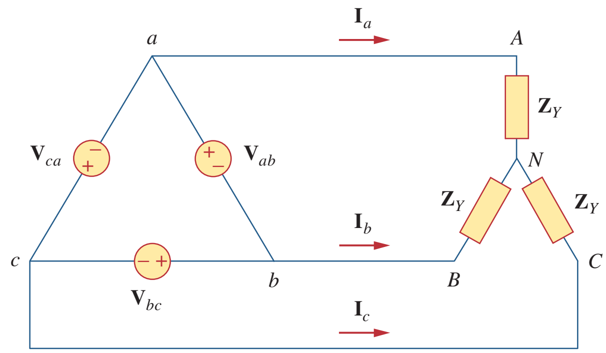


Figure 12.18
A balanced Δ-Y connection.

But $\mathbf{I}_b$ lags $\mathbf{I}_a$ by 120°, since we assumed the *abc* sequence; that is, $\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$. Hence,

$$\begin{aligned}\mathbf{I}_a - \mathbf{I}_b &= \mathbf{I}_a(1 - 1\angle{-120^\circ}) \\ &= \mathbf{I}_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \mathbf{I}_a\sqrt{3}\angle{30^\circ}\end{aligned} \tag{12.36}$$

Substituting Eq. (12.36) into Eq. (12.35) gives

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle{-30^\circ}}{\mathbf{Z}_Y} \tag{12.37}$$

From this, we obtain the other line currents $\mathbf{I}_b$ and $\mathbf{I}_c$ using the positive phase sequence, i.e., $\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$, $\mathbf{I}_c = \mathbf{I}_a\angle{+120^\circ}$. The phase currents are equal to the line currents.

Another way to obtain the line currents is to replace the delta-connected source with its equivalent wye-connected source, as shown in Fig. 12.19. In Section 12.3, we found that the line-to-line voltages of a wye-connected source lead their corresponding phase voltages by 30°. Therefore, we obtain each phase voltage of the equivalent wye-connected source by dividing the corresponding line voltage of the delta-connected source by $\sqrt{3}$ and shifting its phase by −30°. Thus, the equivalent wye-connected source has the phase voltages

$$\mathbf{V}_{an} = \frac{V_p}{\sqrt{3}}\angle{-30^\circ}$$
$$\mathbf{V}_{bn} = \frac{V_p}{\sqrt{3}}\angle{-150^\circ}, \qquad \mathbf{V}_{cn} = \frac{V_p}{\sqrt{3}}\angle{+90^\circ} \tag{12.38}$$

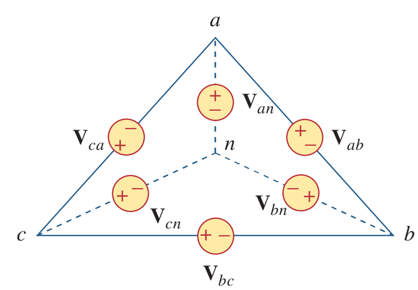


Figure 12.19
Transforming a Δ-connected source to an equivalent Y-connected source.

If the delta-connected source has source impedance $\mathbf{Z}_s$ per phase, the equivalent wye-connected source will have a source impedance of $\mathbf{Z}_s/3$ per phase, according to Eq. (9.69).

Once the source is transformed to wye, the circuit becomes a wye-wye system. Therefore, we can use the equivalent single-phase circuit shown in Fig. 12.20, from which the line current for phase *a* is

$$\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle{-30^\circ}}{\mathbf{Z}_Y} \tag{12.39}$$

which is the same as Eq. (12.37).

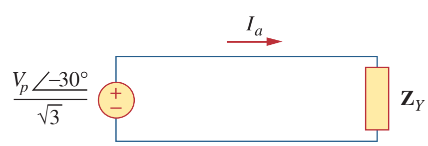


Figure 12.20
The single-phase equivalent circuit.

Alternatively, we may transform the wye-connected load to an equivalent delta-connected load. This results in a delta-delta system, which can be analyzed as in Section 12.5. Note that

$$\mathbf{V}_{AN} = \mathbf{I}_a\mathbf{Z}_Y = \frac{V_p}{\sqrt{3}}\angle{-30^\circ} \tag{12.40}$$
$$\mathbf{V}_{BN} = \mathbf{V}_{AN}\angle{-120^\circ}, \qquad \mathbf{V}_{CN} = \mathbf{V}_{AN}\angle{+120^\circ}$$

As stated earlier, the delta-connected load is more desirable than the wye-connected load. It is easier to alter the loads in any one phase of the delta-connected loads, as the individual loads are connected directly across the lines. However, the delta-connected source is hardly used in practice because any slight imbalance in the phase voltages will result in unwanted circulating currents.

Table 12.1 presents a summary of the formulas for phase currents and voltages and line currents and voltages for the four connections. Students are advised not to memorize the formulas but to understand how they are derived. The formulas can always be

TABLE 12.1

Summary of phase and line voltages/currents for balanced three-phase systems.[1]

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle{0^\circ}$	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle{30^\circ}$
	$\mathbf{V}_{bn} = V_p\angle{-120^\circ}$	$\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle{-120^\circ}$
	$\mathbf{V}_{cn} = V_p\angle{+120^\circ}$	$\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle{+120^\circ}$
	Same as line currents	$\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$
		$\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$
		$\mathbf{I}_c = \mathbf{I}_a\angle{+120^\circ}$
Y-Δ	$\mathbf{V}_{an} = V_p\angle{0^\circ}$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle{30^\circ}$
	$\mathbf{V}_{bn} = V_p\angle{-120^\circ}$	$\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle{-120^\circ}$
	$\mathbf{V}_{cn} = V_p\angle{+120^\circ}$	$\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle{+120^\circ}$
	$\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle{-30^\circ}$
	$\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$
	$\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle{+120^\circ}$
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle{0^\circ}$	Same as phase voltages
	$\mathbf{V}_{bc} = V_p\angle{-120^\circ}$	
	$\mathbf{V}_{ca} = V_p\angle{+120^\circ}$	
	$\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$	$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle{-30^\circ}$
	$\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$	$\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$
	$\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	$\mathbf{I}_c = \mathbf{I}_a\angle{+120^\circ}$
Δ-Y	$\mathbf{V}_{ab} = V_p\angle{0^\circ}$	Same as phase voltages
	$\mathbf{V}_{bc} = V_p\angle{-120^\circ}$	
	$\mathbf{V}_{ca} = V_p\angle{+120^\circ}$	
	Same as line currents	$\mathbf{I}_a = \dfrac{V_p\angle{-30^\circ}}{\sqrt{3}\mathbf{Z}_Y}$
		$\mathbf{I}_b = \mathbf{I}_a\angle{-120^\circ}$
		$\mathbf{I}_c = \mathbf{I}_a\angle{+120^\circ}$

[1] Positive or abc sequence is assumed.

obtained by directly applying KCL and KVL to the appropriate three-phase circuits.

Example 12.5

A balanced Y-connected load with a phase impedance of $40 + j25\ \Omega$ is supplied by a balanced, positive sequence Δ-connected source with a line voltage of 210 V. Calculate the phase currents. Use $\mathbf{V}_{ab}$ as a reference.

Solution:
The load impedance is

$$\mathbf{Z}_Y = 40 + j25 = 47.17\underline{/32^\circ}\ \Omega$$

and the source voltage is

$$\mathbf{V}_{ab} = 210\underline{/0^\circ}\text{ V}$$

When the Δ-connected source is transformed to a Y-connected source,

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\underline{/-30^\circ} = 121.2\underline{/-30^\circ}\text{ V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\underline{/-30^\circ}}{47.12\underline{/32^\circ}} = 2.57\underline{/-62^\circ}\text{ A}$$
$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 2.57\underline{/-178^\circ}\text{ A}$$
$$\mathbf{I}_c = \mathbf{I}_a\underline{/120^\circ} = 2.57\underline{/58^\circ}\text{ A}$$

which are the same as the phase currents.

Practice Problem 12.5

In a balanced Δ-Y circuit, $\mathbf{V}_{ab} = 240\underline{/15^\circ}$ and $\mathbf{Z}_Y = (12 + j15)\ \Omega$. Calculate the line currents.

Answer: $7.21\underline{/-66.34^\circ}$ A, $7.21\underline{/+173.66^\circ}$ A, $7.21\underline{/53.66^\circ}$ A.

12.7 Power in a Balanced System

Let us now consider the power in a balanced three-phase system. We begin by examining the instantaneous power absorbed by the load. This requires that the analysis be done in the time domain. For a Y-connected load, the phase voltages are

$$v_{AN} = \sqrt{2}V_p \cos\omega t, \qquad v_{BN} = \sqrt{2}V_p \cos(\omega t - 120^\circ)$$
$$v_{CN} = \sqrt{2}V_p \cos(\omega t + 120^\circ) \tag{12.41}$$

where the factor $\sqrt{2}$ is necessary because V_p has been defined as the rms value of the phase voltage. If $\mathbf{Z}_Y = Z\underline{/\theta}$, the phase currents lag behind their corresponding phase voltages by θ. Thus,

$$i_a = \sqrt{2}I_p \cos(\omega t - \theta), \qquad i_b = \sqrt{2}I_p \cos(\omega t - \theta - 120^\circ)$$
$$i_c = \sqrt{2}I_p \cos(\omega t - \theta + 120^\circ) \tag{12.42}$$