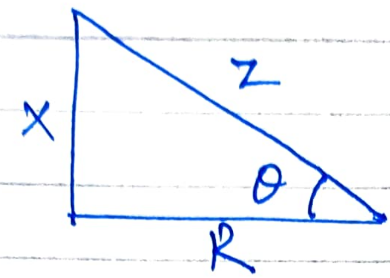
Impedance Triangle



$\cos\theta = \frac{R}{Z}$

(power factor)

$i = \left|\frac{V}{|Z|}\right| \angle -\theta_Z$ ← cosine of this angle also gives the power factor.

Resonance :- (series RLC)

$$Z = R + j(X_L - X_C)$$

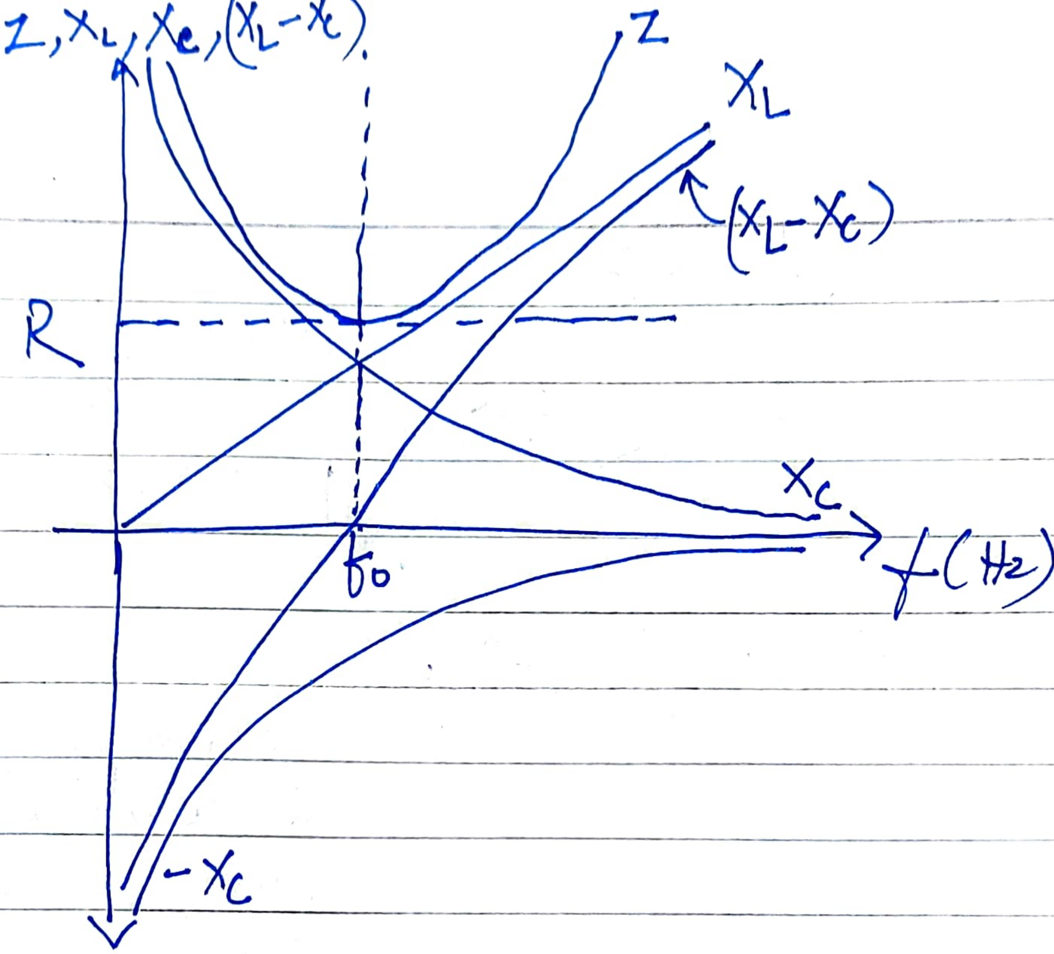
when $X_L = X_C \Rightarrow X_L - X_C = 0$
(net reactance $= 0$)

then $Z = R$

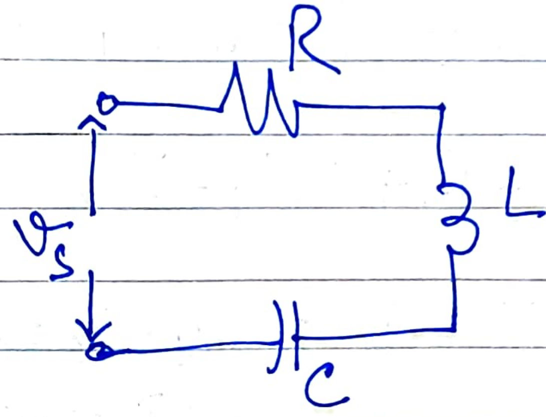
$\Downarrow$

the circuit behaves as a pure resistor.

At this condition, the impedance of the ckt. is minimum and equals R.



for series RLC ckt:-



$$v_s = V_m \sin(\omega t)$$

$$X_L = j\omega L$$

$$X_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

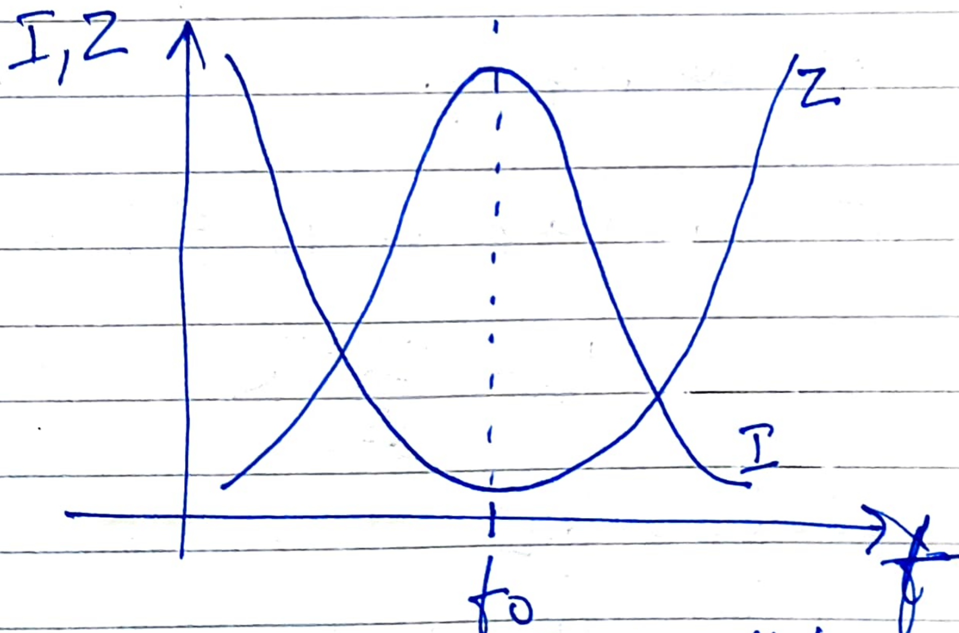
$$Z = \sqrt{R^2 + j\left(\omega L - \frac{1}{\omega C}\right)}$$

at resonance :-

$$X_L = X_C$$

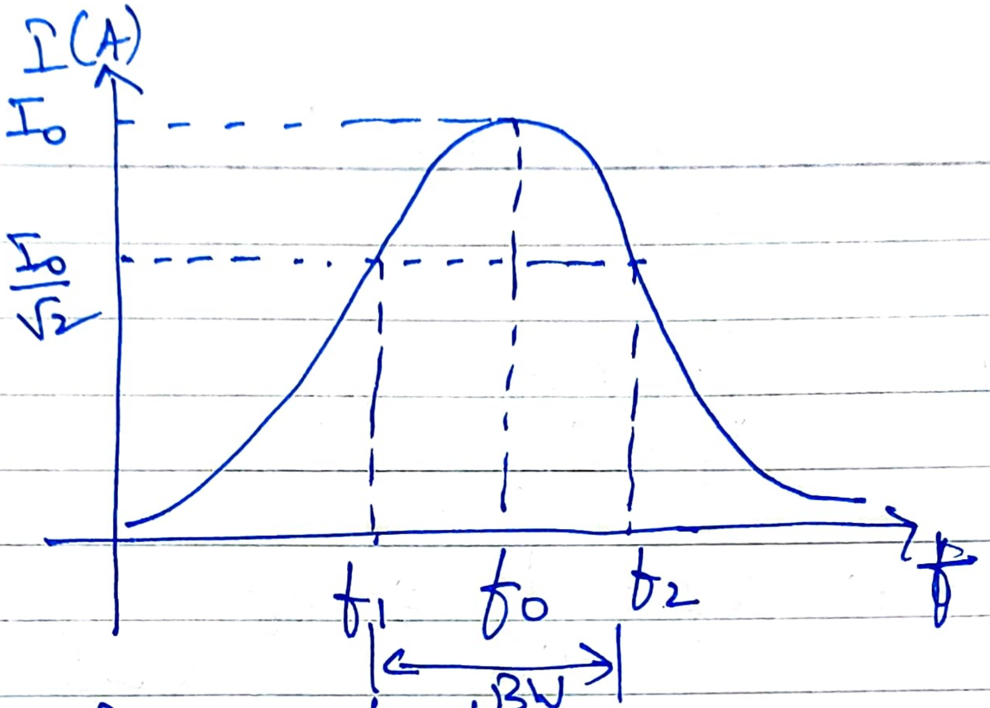
$$\Rightarrow \omega_0 L = \frac{1}{\omega C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow \boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}} \text{ Hz}$$



Since $Z = R$ is min. I will be max.

so, at f_0 $\boxed{I_0 = \frac{V}{R}}$



<u>Half Power-pt. frequencies:-</u>

At f_0 $P_0 = I_0^2 R$

~~At half freq.~~

for half of $P_0 \Rightarrow P_h = \frac{I_0^2 R}{2}$

$$P_h = \left(\frac{I_0}{\sqrt{2}}\right)^2 R$$

~~At f_1 & f_2~~

For $f_1 \leq f \leq f_2$ current will be

~~more than or equal~~ atleast $\frac{I_0}{\sqrt{2}}$

and the power will be more than $\frac{P_0}{2}$.

$$\omega_1 = \omega_0 - \frac{R}{2L}$$

$$\omega_2 = \omega_0 + \frac{R}{2L}$$

$$\Rightarrow f_1 = f_0 - \frac{R}{4\pi L} \quad \text{— A}$$

$$f_2 = f_0 + \frac{R}{4\pi L} \quad \text{— B}$$

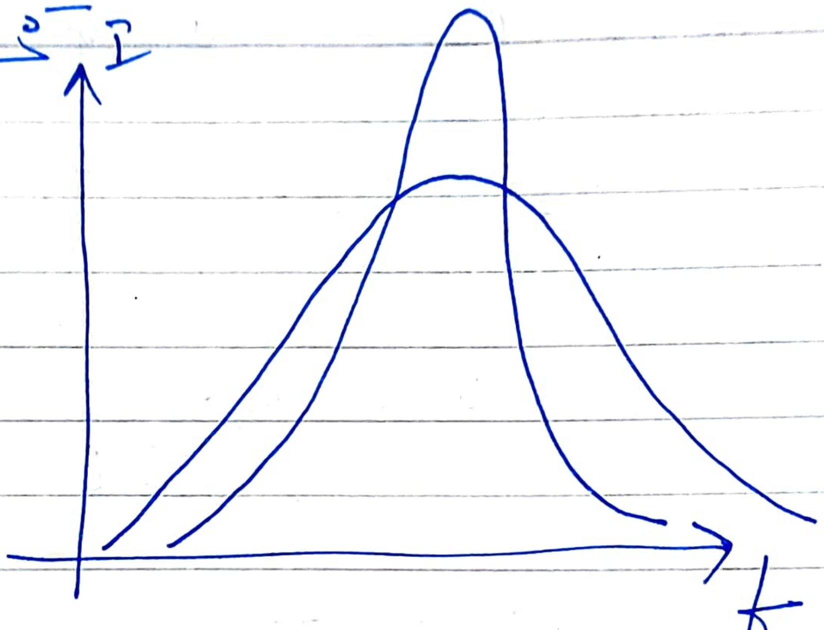
Band width $BW = f_2 - f_1 = B$

$$\boxed{B = \frac{R}{2\pi L} = f_2 - f_1}$$

Adding A & B

$$\boxed{f_0 = \frac{f_1 + f_2}{2}} \quad \text{— C}$$

Q-factor :-



The sharpness of the freq. response curve is measured by the Quality factor.

$$Q = \frac{\text{Energy stored}}{\text{Energy dissipated}} = \frac{\text{Power Stored}}{\text{Power dissipated}}$$

$$= \frac{I_0^2 X_L}{I_0^2 R} = \frac{I_0^2 X_C}{I_0^2 R}$$

$$Q = \frac{X_L}{R} = \frac{\omega_0 L}{R}$$

$$Q = \frac{X_C}{R} = \frac{1}{\omega_0 C R}$$

for series resonance

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow Q = \frac{1}{\sqrt{LC}}\,\frac{L}{R}$$

$$\boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

$$\cancel{f = \frac{1}{2\pi\sqrt{LC}}} \qquad Q = \frac{f_0}{B}$$

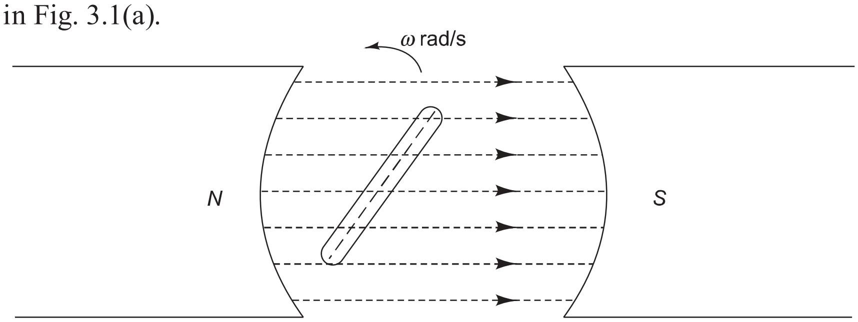
$$\cancel{\frac{1}{2\pi L}} \qquad \frac{1}{\cancel{2\pi}}\,\frac{1}{\sqrt{LC}}\cdot\frac{\cancel{2\pi} L}{R}$$

$$\boxed{Q = \frac{f_0}{B}}$$

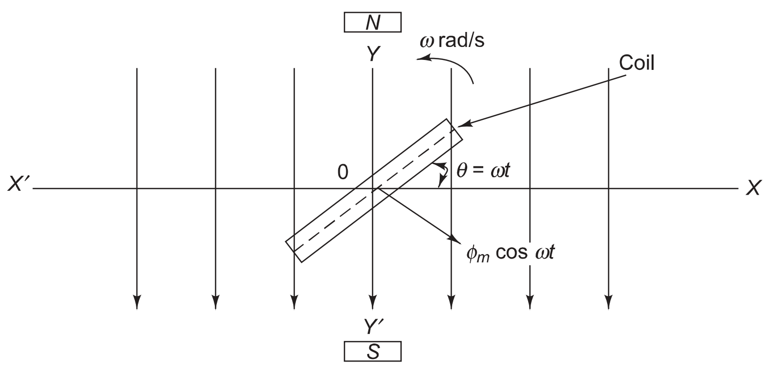
3.1 GENERATION OF ALTERNATING VOLTAGES

An alternating voltage can be generated either by rotating a coil in a stationary magnetic field or by a rotating a magnetic field within a stationary coil. In both the cases, the magnetic field is cut by the conductors or coils and an emf is induced in the coil according to Faraday's laws of electromagnetic induction. The magnitude of the induced emf depends upon the number of turns of the coil, the strength of the magnetic field and the speed at which the coil or magnetic field rotates.

Consider a rectangular coil of N turns of area A m^2 and rotating in anti-clockwise direction with angular velocity of ω radians per second in a uniform magnetic field as shown in Fig. 3.1(a).



(a)



(b)

Fig. 3.1 *Generation of alternating voltage*

Let ϕ_m be the maximum flux cutting the coil when its axis coincides with the XX' axis (reference position of the coil). Thus when the coil is along XX', the flux linking with it is maximum, i.e., ϕ_m. When the coil is along YY', i.e., parallel to the lines of flux, the flux linking with it is zero.

The coil rotates through an angle $\theta = \omega t$ at any instant t.

At this instant, the flux linking with the coil is

$$\phi = \phi_m \cos \omega t$$

According to Faraday's laws of electromagnetic induction,

$$e = -N\frac{d\phi}{dt}$$
$$= -N\frac{d}{dt}(\phi_m \cos\omega t)$$
$$= N\,\phi_m\,\omega \sin \omega t$$
$$= E_m \sin \omega t$$

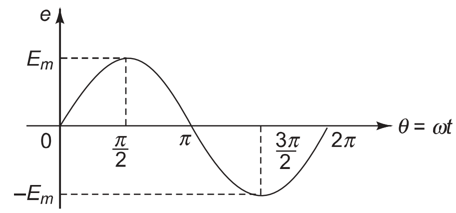


Fig. 3.2 *Sinusoidal waveform*

where $E_m = N\,\phi_m\omega$

$= \text{maximum value of induced emf}$

When $\omega t = 0, \quad \sin \omega t = 0, \quad e = 0$

When $\omega t = \frac{\pi}{2}, \quad \sin \frac{\pi}{2} = 1 \quad e = E_m$

If the induced emf is plotted against time, a sinusoidal waveform is obtained.

3.2 TERMS RELATED TO ALTERNATING QUANTITIES

[May 2015]

Waveform A waveform is a graph in which the instantaneous value of any quantity is plotted against time. Figure 3.3 shows a few waveforms.

Cycle One complete set of positive and negative values of an alternating quantity is termed a cycle.

Frequency The number of cycles per second of an alternating quantity is known as its frequency. It is denoted by f and is measured in hertz (Hz) or cycles per second (c/s).

Time Period The time taken by an alternating quantity to complete one cycle is called its time period. It is denoted by T and is measured in seconds.

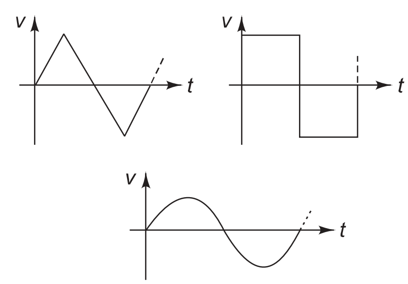


Fig. 3.3 *Alternating waveforms*

$$T = \frac{1}{f}$$

Amplitude The maximum positive or negative value of an alternating quantity is called the amplitude.

Phase The phase of an alternating quantity is the time that has elapsed since the quantity has last passed through zero point of reference.

Phase Difference This term is used to compare the phases of two alternating quantities. Two alternating quantities are said to be in phase when they reach their maximum and zero values at the same time. Their maximum value may be different in magnitude.

A leading alternating quantity is one which reaches its maximum or zero value earlier compared to the other quantity.

A lagging alternating quantity is one which attains its maximum or zero value later than the other quantity.

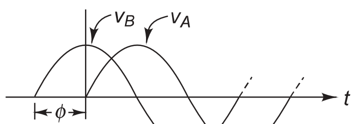


Fig. 3.4 *Phase difference*

A plus (+) sign, when used in connection with the phase difference, denotes 'lead' whereas a minus (–) sign denotes 'lag'.

$$v_A = V_m \sin \omega t$$

$$v_B = V_m \sin (\omega t + \phi)$$

Here, the quantity B leads A by a phase angle ϕ.

3.3 ROOT MEAN SQUARE (RMS) OR EFFECTIVE VALUE

Normally, the current is measured by the amount of work it will do or the amount of heat it will produce. Hence, rms or effective value of alternating current is defined as that value of steady current (direct current) which will do the same amount of work in the same time or would produce the same heating effect as when the alternating current is applied for the same time.

Figure 3.5 shows the positive half cycle of a non-sinusoidal alternating current waveform. The waveform is divided in m equal intervals with the instantaneous currents, these intervals being $i_1, i_2, ..., i_m$. This waveform is applied to a circuit consisting of a resistance of R ohms. Then work done in different intervals will be $\left(i_1^2 R \times \frac{t}{m}\right), \left(i_2^2 R \times \frac{t}{m}\right), ..., \left(i_m^2 R \times \frac{t}{m}\right)$ joules.

Thus, the total work done in t seconds on applying an alternating current waveform to a resistance $R = \frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m} \times Rt$ joules

Let I be the value of the direct current that while flowing through the same resistance does the same amount of work in the same time t. Then

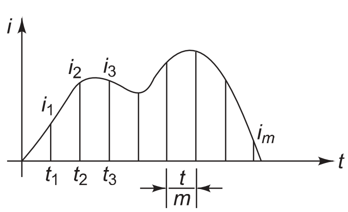


Fig. 3.5 *Mid-ordinate method*

$$I^2Rt = \frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m} \times Rt$$

$$I^2 = \frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m}$$

Hence, rms value of the alternating current is given by

$$I_{\text{rms}} = \sqrt{\frac{i_1^2 + i_2^2 + \ldots + i_m^2}{m}} = \sqrt{\text{Mean value of} (i)^2}$$

The rms value of any current $i(t)$ over the specified interval t_1 to t_2 is expressed mathematically as

$$I_{\text{rms}} = \sqrt{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} i^2(t)\,dt}$$

The rms value of an alternating current is of considerable importance in practice because the ammeters and voltmeters record the rms value of alternating current and voltage respectively.

3.3.1 RMS Value of Sinusoidal Waveform

$$v = V_m \sin\theta \qquad 0 < \theta < 2\pi$$

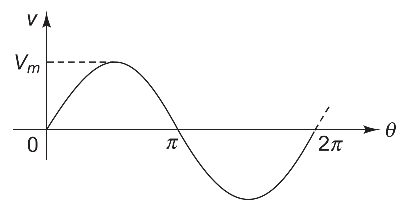


Fig. 3.6 *Sinusoidal waveform*

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_m^2 \sin^2\theta\,d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{2\pi} \sin^2\theta\,d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{2\pi}{2} - 0 - 0 + 0\right]}$$

$$= \sqrt{\frac{V_m^2}{2}}$$

$$= \frac{V_m}{\sqrt{2}}$$

$$= 0.707\,V_m$$

Crest or Peak or Amplitude Factor It is defined as the ratio of maximum value to rms value of the given quantity.

$$\text{Peak factor } (k_p) = \frac{\text{Maximum value}}{\text{rms value}}$$

3.4 AVERAGE VALUE

The average value of an alternating quantity is defined as the arithmetic mean of all the values over one complete cycle.

In case of a symmetrical alternating waveform (whether sinusoidal or non-sinusoidal), the average value over a complete cycle is zero. Hence, in such a case, the average value is obtained over half the cycle only.

Referring to Fig. 3.5, the average value of the current is given by

$$I_{\text{avg}} = \frac{i_1 + i_2 + \ldots + i_m}{m}$$

The average value of any current $i(t)$ over the specified interval t_1 to t_2 is expressed mathematically as

$$I_{\text{avg}} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} i(t)\,dt$$

3.4.1 Average Value of Sinusoidal Waveform [Dec 2013]

$$v = V_m \sin\theta \qquad 0 < \theta < 2\pi$$

Since this is a symmetrical waveform, the average value is calculated over half the cycle.

$$V_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\; d\theta$$

$$= \frac{V_m}{\pi}\int_0^{\pi} \sin\theta\; d\theta$$

$$= \frac{V_m}{\pi}[-\cos\theta]_0^{\pi}$$

$$= \frac{V_m}{\pi}[1+1]$$

$$= \frac{2V_m}{\pi}$$

$$= 0.637\, V_m$$

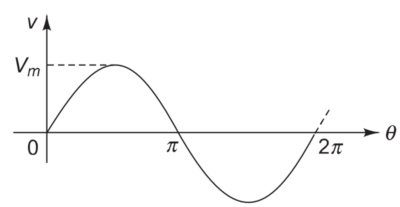


Fig. 3.7 *Sinusoidal waveform*

Form Factor It is defined as the ratio of rms value to the average value of the given quantity.

$$\text{Form factor } (k_f) = \frac{\text{rms value}}{\text{Average value}}$$

Example 1

An alternating current takes 3.375 ms to reach 15 A for the first time after becoming instantaneously zero. The frequency of the current is 40 Hz. Find the maximum value of the alternating current. **[May 2014]**

Solution

$$i = 15 \text{ A}$$
$$t = 3.375 \text{ ms}$$
$$f = 40 \text{ Hz}$$
$$i = I_m \sin 2\pi f t$$
$$15 = I_m \sin (2 \times 180 \times 40 \times 3.375 \times 10^{-3}) \qquad \text{(angle in degrees)}$$
$$15 = I_m \times 0.75$$
$$I_m = 20 \text{ A}$$

Example 2

An alternating current of 50 c/s frequency has a maximum value of 100 A. (i) Calculate its value $\frac{1}{600}$ second and after the instant the current is zero. (ii) In how many seconds after the zero value will the current attain the value of 86.6 A?

Solution

$$f = 50 \text{ c/s}$$
$$I_m = 100 \text{ A}$$

(i) Value of current $\frac{1}{600}$ second after the instant the current is zero

$$i = I_m \sin 2\pi f t$$
$$= 100 \sin \left(2 \times 180 \times 50 \times \frac{1}{600}\right) \qquad \text{(angle in degrees)}$$
$$= 100 \sin (30^\circ)$$
$$= 50 \text{ A}$$

(ii) Time at which current will attain the value of 86.6 A after the zero value

$$i = I_m \sin 2\pi f t$$
$$86.6 = 100 \sin (2 \times 180 \times 50 \times t) \qquad \text{(angle in degrees)}$$
$$\sin (18000\, t) = 0.866$$
$$18000\, t = 60^\circ$$
$$t = \frac{1}{300} \text{ second}$$

Example 3

An alternating current varying sinusoidally with a frequency of 50 c/s has an rms value of 20 A. Write down the equation for the instantaneous value and find this value at (i) 0.0025 s, and (ii) 0.0125 s after passing through zero and increasing positively. (iii) At what time, measured from zero, will the value of the instantaneous current be 14.14 A?

Solution

$$f = 50 \text{ c/s}$$

$$I_{\text{rms}} = 20 \text{ A}$$

$$I_m = I_{\text{rms}} \times \sqrt{2} = 20\sqrt{2} = 28.28 \text{ A}$$

Equation of current, $i = I_m \sin 2\pi f t$

$$= 28.28 \sin (100\pi \times t)$$

$$= 28.28 \sin (100 \times 180 \times t) \qquad \text{(angle in degrees)}$$

(i) Value of current at $t = 0.0025$ second

$$i = 28.28 \sin (100 \times 180 \times 0.0025) \qquad \text{(angle in degrees)}$$

$$= 28.28 \sin (45^\circ)$$

$$= 20 \text{ A}$$

(ii) Value of current at $t = 0.0125$ second

$$i = 28.28 \sin (100 \times 180 \times 0.0125) \qquad \text{(angle in degrees)}$$

$$= 28.28 \sin (225^\circ)$$

$$= -20 \text{ A}$$

(iii) Time at which value of instantaneous current will be 14.14 A

$$i = 28.28 \sin 100\pi t$$

$$14.14 = 28.28 \sin 18000\, t \qquad \text{(angle in degrees)}$$

$$\sin 18000\, t = 0.5$$

$$18000\, t = 30^\circ$$

$$t = 1.66 \text{ ms}$$

Example 4

An alternating current of 60 Hz frequency has a maximum value of 110 A. Calculate (i) time required to reach 90 A after the instant current is zero and increasing positively, and (ii) its value $\frac{1}{600}$ second after the instant current is zero and its value decreasing thereafter.

Solution $f = 60$ Hz

$I_m = 110$ A

(i) Time required to reach 90 A after the instant current is zero and increasing positively.

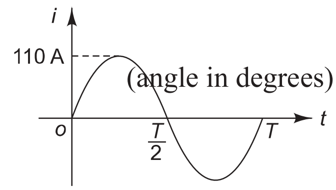


Fig. 3.8

$$i = I_m \sin 2\pi ft$$

$$90 = 110 \sin (2 \times 180 \times 60 \times t) \qquad \text{(angle in degrees)}$$

$$\sin 21600\, t = 0.818$$

$$21600\, t = 54.88°$$

$$t = 2.54 \text{ ms}$$

(ii) Value of current $\frac{1}{600}$ second after the instant current is zero and decreasing thereafter.

From Fig. 3.8,

$$t = \frac{T}{2} + \frac{1}{600}$$

$$= \frac{1}{2f} + \frac{1}{600}$$

$$= \frac{1}{2 \times 60} + \frac{1}{600}$$

$$= 0.01 \text{ s}$$

$$i = I_m \sin 2\pi ft$$

$$= 110 \sin (2 \times 180 \times 60 \times 0.01) \qquad \text{(angle in degrees)}$$

$$= -64.66 \text{ A}$$

Example 5

A sinusoidal wave of 50 Hz frequency has its maximum value of 9.2 A. What will be its value at (i) 0.002 s after the wave passes through zero in the positive direction, and (ii) 0.0045 s after the wave passes through the positive maximum.

Solution $f = 50$ Hz

$I_m = 9.2$ A

(i) Value of current at 0.002 s after the wave passes through zero in the positive direction

$$i = I_m \sin 2\pi ft$$

$$= 9.2 \sin (2 \times 180 \times 50 \times 0.002) \qquad \text{(angle in degrees)}$$

$$= 5.41 \text{ A}$$

(ii) Value of current 0.0045 s after the wave passes through the positive maximum

From Fig. 3.9,

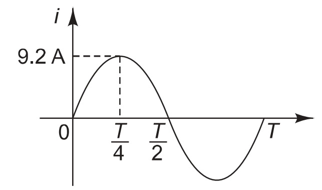


Fig. 3.9

$$t = \frac{T}{4} + 0.0045$$

$$= \frac{1}{4f} + 0.0045$$

$$= \frac{1}{4 \times 50} + 0.0045$$

$$= 9.5 \text{ ms}$$

$$i = I_m \sin 2\pi ft$$

$$= 9.2 \sin (2 \times 180 \times 50 \times 9.5 \times 10^{-3}) \quad \text{(angle in degrees)}$$

$$= 1.44 \text{ A}$$

Example 6

An alternating current varying sinusoidally with a frequency of 50 Hz has an rms value of current of 20 A. At what time measured from negative maximum value will the instantaneous current be $10\sqrt{2}$ A?

Solution

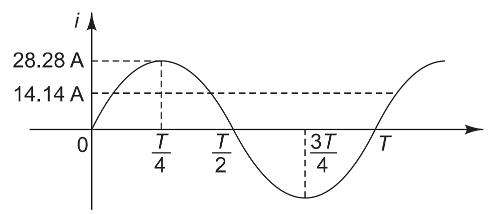


Fig. 3.10

$$f = 50 \text{ Hz}$$

$$I_{\text{rms}} = 20 \text{ A}$$

(i) Time at which instantaneous current will be $10\sqrt{2}$ A

$$i = 10\sqrt{2} \text{ A} = 14.14 \text{ A}$$

$$I_m = I_{\text{rms}} \times \sqrt{2} = 20\sqrt{2} = 28.28 \text{ A}$$

$$i = I_m \sin 2\pi ft$$

$$14.14 = 28.28 \sin (2 \times 180 \times 50 \times t) \quad \text{(angle in degrees)}$$

$$0.5 = \sin (18000\, t)$$

$$18000\, t = 30°$$

$$t = 1.67 \text{ ms}$$

(ii) Time, measured from negative maximum value, at which instantaneous current will be $10\sqrt{2}$ A

$$t = \frac{T}{4} + 1.67 \times 10^{-3}$$

$$= \frac{1}{4f} + 1.67 \times 10^{-3}$$

$$= \frac{1}{4 \times 50} + 1.67 \times 10^{-3}$$
$$= 6.67 \text{ ms}$$

Example 7

An alternating voltage is represented by v = 141.4 sin 377t. Find (i) max-value (ii) frequency (iii) time period. **[May 2016]**

Solution $v = 141.4 \sin 377\, t$

(i) Maximum value

Comparing with the equation $v = V_m \sin 2\pi ft$,

$$V_m = 141.4 \text{ V}$$

(ii) Frequency

$$2\pi f = 377$$
$$f = \frac{377}{2\pi} = 60 \text{ Hz}$$

(iii) Time period

$$T = \frac{1}{f} = \frac{1}{60} = 16.67 \text{ ms}$$

Example 8

An alternating voltage is given by v = 141.4sin 314t. Find (i) frequency, (ii) rms value, (iii) average value, and (iv) instantaneous value of voltage, when t is 3 ms. **[Dec 2012]**

Solution $v = 141.4 \sin 314\, t$

(i) Frequency

Comparing with the equation $v = V_m \sin 2\pi ft$,

$$2\pi f = 314$$
$$f = \frac{314}{2\pi} = 49.97 \text{ Hz}$$

(ii) rms value

$$V_m = 141.4 \text{ V}$$
$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} = 99.98 \text{ V}$$

(iii) Average value

$$V_{\text{avg}} = \frac{2V_m}{\pi} = \frac{2 \times 141.4}{\pi} = 90.02 \text{ V}$$

(iv) Instantaneous value of the voltage at $t = 3$ ms

$$v = 141.4 \sin 314 \times 3 \times 10^{-3} = 114.36 \text{ V}$$

Example 9

An alternating current is given by $i = 14.14 \sin 377\, t$. Find (i) rms value of the current, (ii) frequency, (iii) instantaneous value of the current when $t = 3$ ms, and (iv) time taken by the current to reach 10 A for first time after passing through zero.

Solution $i = 14.14 \sin 377\, t$

(i) The rms value of the current

$$I_{\text{rms}} = \frac{I_m}{\sqrt{2}} = \frac{14.14}{\sqrt{2}} = 10 \text{ A}$$

(ii) Frequency

$$2\pi f = 377$$

$$f = \frac{377}{2\pi} = 60 \text{ Hz}$$

(iii) Instantaneous value of the current when $t = 3$ ms

$$i = 14.14 \sin (377 \times 3 \times 10^{-3}) \qquad \text{(angle in radians)}$$

$$= 12.79 \text{ A}$$

(iv) Time taken by the current to reach 10 A for the first time after passing through zero

$$i = 14.14 \sin 377\, t \qquad \text{(angle in radians)}$$

$$10 = 14.14 \sin 377\, t$$

$$\sin 377\, t = 0.707$$

$$377\, t = 0.79$$

$$t = 2.084 \text{ ms}$$

Example 10

An alternating current varying sinusoidally at 50 Hz has its rms value of 10 A. Write down an equation for the instantaneous value of the current. Find the value of the current at (i) 0.0025 second after passing through the positive maximum value, and (ii) 0.0075 second after passing through zero value and increasing negatively.

Solution $f = 50$ Hz

$$I_{\text{rms}} = 10 \text{ A}$$

(i) Equation for instantaneous value of the current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} = 14.14 \text{ A}$$

$$i = I_m \sin 2\pi f t$$

$$= 14.14 \sin (2 \times 180 \times 50 \times t) \qquad \text{(angle in degrees)}$$

$$= 14.14 \sin (18000\, t)$$

(ii) Value of the current at 0.0025 s after passing through the positive maximum value

From Fig. 3.11(a),

$$t = \frac{T}{4} + 0.0025$$

$$= \frac{1}{4f} + 0.0025$$

$$= \frac{1}{4 \times 50} + 0.0025$$

$$= 7.5 \text{ ms}$$

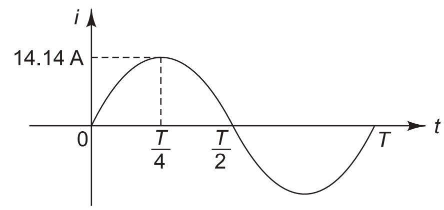


Fig. 3.11(a)

$$i = 14.14 \sin (18000 \times 7.5 \times 10^{-3}) \qquad \text{(angle in degrees)}$$

$$= 10 \text{ A}$$

(ii) Value of the current 0.0075 s after passing through zero value and increasing negatively

From Fig. 3.11(b),

$$t = \frac{T}{2} + 0.0075$$

$$= \frac{1}{2f} + 0.0075$$

$$= \frac{1}{2 \times 50} + 0.0075$$

$$= 17.5 \text{ ms}$$

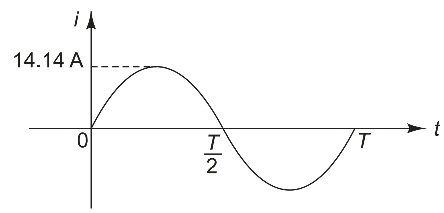


Fig. 3.11(b)

$$i = 14.14 \sin (18000 \times 17.5 \times 10^{-3}) \qquad \text{(angle in degrees)}$$

$$= -10 \text{ A}$$

Example 11

Draw a neat sketch in each case of the waveform and write expressions of instantaneous value for the following:

(i) Sinusoidal current of amplitude 10 A, 50 Hz passing through its zero value at $\omega t = \frac{\pi}{3}$ *and increasing positively*

(ii) Sinusoidal current of amplitude 8 A, 50 Hz passing through its zero value at $\omega t = -\frac{\pi}{6}$ *and increasing positively.*

Solution (i) The current waveform is lagging in nature.

$$i = I_m \sin (2\pi ft - \phi)$$

$$= 10 \sin \left(2\pi \times 50 \times t - \frac{\pi}{3}\right)$$

$$= 10 \sin \left(100\pi t - \frac{\pi}{3}\right)$$

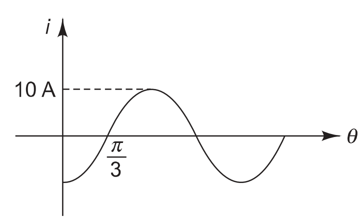


Fig. 3.12(a)

(ii) The current waveform is leading in nature.

$$i = I_m \sin(2\pi ft + \phi)$$
$$= 8 \sin\left(2\pi \times 50 \times t + \frac{\pi}{6}\right)$$
$$= 8 \sin\left(100\pi t + \frac{\pi}{6}\right)$$

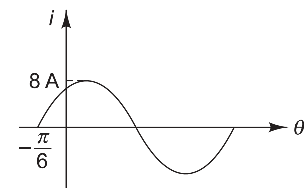


Fig. 3.12(b)

Example 12

The instantaneous current is given by $I = 7.071 \sin\left(157.08t - \frac{\pi}{4}\right)$. Find its effective value, periodic time and the instant at which it reaches its positive maximum value. Sketch the waveform from t = 0 over one complete cycle.

Solution $\quad i = 7.071 \sin\left(157.08t - \frac{\pi}{4}\right)$

(i) Effective value

$$I_{\text{eff}} = I_{\text{rms}} = \frac{I_m}{\sqrt{2}} = \frac{7.071}{\sqrt{2}} = 5 \text{ A}$$

(ii) Periodic time

$$2\pi f = 157.08$$
$$f = 25 \text{ Hz}$$
$$T = \frac{1}{f} = \frac{1}{25} = 0.04 \text{ s}$$

(iii) Instant at which the current reaches its positive maximum value, i.e., $i = 7.071$ A

$$7.071 = 7.071 \sin\left(157.08t - \frac{\pi}{4}\right) \quad \text{(angle in radians)}$$
$$1 = \sin(157.08t - 0.785)$$
$$1.5708 = 157.08t - 0.785$$
$$t = 0.015 \text{ s}$$

(iv) The waveform is shown in Fig. 3.13.

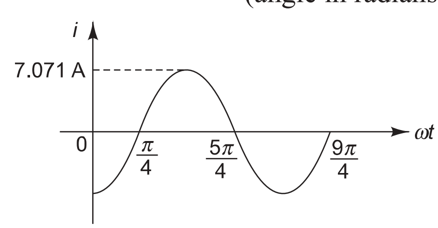


Fig. 3.13

Example 13

A 60 Hz sinusoidal current has an instantaneous value of 7.07 A at t = 0 and rms value of $10\sqrt{2}$ A. Assuming the current wave to enter positive half at t = 0, determine (i) expression for instantaneous current, (ii) magnitude of the current at t = 0.0125 second, and (iii) magnitude of the current at t = 0.025 second after t = 0.

Solution $\quad f = 60$ Hz

$$i(0) = 7.07 \text{ A}$$
$$I_{\text{rms}} = 10\sqrt{2} \text{ A}$$

(i) Expression for instantaneous current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} \times \sqrt{2} = 20 \text{ A}$$

Since $i = 7.07$ A at $t = 0$, the current is leading in nature.

At $t = 0$,

$$i = I_m \sin(2\pi ft + \phi)$$
$$7.07 = 20 \sin(2\pi \times 60 \times 0 + \phi)$$
$$\phi = 20.7°$$
$$i = 20 \sin(120\pi t + 20.7°)$$

(ii) Magnitude of current at $t = 0.0125$ second

$$i = 20 \sin(120 \times 180 \times 0.0125 + 20.7°) \qquad \text{(angle in degrees)}$$
$$= -18.7 \text{ A}$$

(iii) Magnitude of current at $t = 0.025$ second after $t = 0$

Time corresponding to a phase shift of 20.7°

$$= \frac{20.7°}{360°} \times T$$
$$= \frac{20.7°}{360°} \times \frac{1}{f}$$
$$= \frac{20.7°}{360°} \times \frac{1}{60}$$
$$= 0.958 \text{ ms}$$

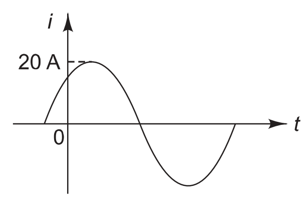


Fig. 3.14

Time 0.025 second after $t = 0$

$$t = 0.958 \times 10^{-3} + 0.025$$
$$= 25.958 \text{ ms}$$
$$i = 20 \sin(120 \times 180 \times 25.958 \times 10^{-3} + 20.7°) \qquad \text{(angle in degrees)}$$
$$= -13.22 \text{ A}$$

Example 14

A 50 Hz sinusoital voltage applied to a single-phase circuit has an rms value of 200 V. Its value at t = 0 is $(\sqrt{2} \times 200)$ *V positive. The current drawn by the circuit is 5 A (rms) and lags behind the voltage by one-sixth of a cycle. Write the expressions for the instantaneous values of voltage and current. Sketch their waveforms and find their values at t = 0.0125 second.*

Solution

$$f = 50 \text{ Hz}$$
$$V_{\text{rms}} = 200 \text{ V}$$
$$I_{\text{rms}} = 5 \text{ A}$$

$$v(0) = \sqrt{2} \times 200 = 282.84 \text{ V}$$

(i) Instantaneous value of voltage

$$V_m = V_{\text{rms}} \times \sqrt{2} = 200\sqrt{2} = 282.84 \text{ V}$$

$$v = V_m \sin(2\pi f t + \phi)$$

At $t = 0$,

$$282.84 = 282.84 \sin(0 + \phi)$$

$$\sin\phi = 1$$

$$\phi = 90°$$

$$v = 282.84 \sin(2\pi \times 50 \times t + 90°)$$

$$= 282.84 \sin(100\,\pi t + 90°)$$

(ii) Instantaneous value of current

The current lags behind the voltage by one-sixth of a cycle.

$$\phi = \frac{1}{6} \times 360 = 60°$$

$$I_m = I_{\text{rms}} \times \sqrt{2} = 5\sqrt{2} = 7.07 \text{ A}$$

$$i = I_m \sin(2\pi f t + 90° - 60°)$$

$$= 7.07 \sin(2\pi \times 50 \times t + 30°)$$

$$= 7.07 \sin(100\pi t + 30°)$$

(iii) Voltage and current waveforms are shown in Fig. 3.15.

(iv) Value of voltage at $t = 0.0125$ s

$$v = 282.84 \sin(100 \times 180 \times 0.0125 + 90°)$$

(angle in degrees)

$$= 200 \text{ V}$$

(v) Value of current at $t = 0.0125$ s

$$i = 7.07 \sin(100 \times 180 \times 0.0125 + 30°)$$

(angle in degrees)

$$= -6.83 \text{ A}$$

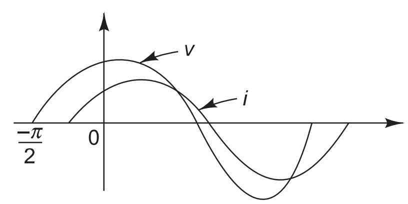


Fig. 3.15

Example 15

At the instant $t = 0$, the instantaneous value of a 50 Hz sinusoidal current is 5 A and increases in magnitude further. Its rms value is 10 A. (i) Write the expression for its instantaneous value. (ii) Find the current at $t = 0.01$ s and $t = 0.015$ s. (iii) Sketch the waveform indicating these values.

Solution

$$f = 50 \text{ Hz}$$

$$i(0) = 5 \text{ A}$$

$$I_{\text{rms}} = 10 \text{ A}$$

(i) Expression for instantaneous value of current

$$I_m = I_{\text{rms}} \times \sqrt{2} = 10\sqrt{2} = 14.14 \text{ A}$$

Since $i = 5$ A at $t = 0$, the current is leading in nature.

$$i = I_m \sin (2\pi f t + \phi)$$

At $t = 0$,

$$5 = 14.14 \sin (2 \times 180 \times 50 \times 0 + \phi) \qquad \text{(angle in degrees)}$$

$$5 = 14.14 \sin \phi$$

$$\phi = 20.7°$$

$$i = 14.14 \sin (100\pi t + 20.7°)$$

(ii) Current at $t = 0.01$ s

$$i = 14.14 \sin (100 \times 180 \times 0.01 + 20.7°) \qquad \text{(angle in degrees)}$$

$$= -5 \text{ A}$$

(iii) Current at $t = 0.015$ s

$$i = 14.14 \sin (100 \times 180 \times 0.015 + 20.7°) \qquad \text{(angle in degrees)}$$

$$= -13.23 \text{ A}$$

(iv) Waveform

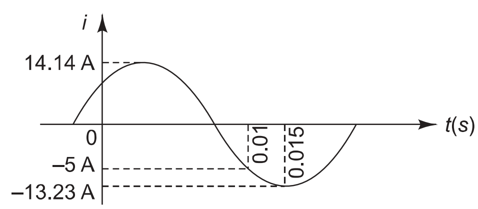


Fig. 3.16

Example 16

In a certain circuit supplied from 50 Hz mains, the potential difference has a maximum value of 500 V and the current has a maximum value of 10 A. At the instant t = 0, the instantaneous values of potential difference and current are 400 V and 4 A respectively, both increasing in the positive direction. State expressions for instantaneous values of potential difference and current at time t. Calculate the instantaneous values at time t = 0.015 second. Find the phase angle between potential difference and current.

Solution

$$f = 50 \text{ Hz}$$

$$V_m = 500 \text{ V}$$

$$I_m = 10 \text{ A}$$

$$v(0) = 400 \text{ V}$$

$$i(0) = 4 \text{ A}$$

(i) Expression for instantaneous values of potential difference and current.
Since $v = 400$ V and $i = 4$ A at $t = 0$, the voltage and current waveforms are leading in nature.

(a) $v = V_m \sin(2\pi ft + \phi_1)$

At $t = 0$,

$$400 = 500 \sin(2\pi \times 50 \times 0 + \phi_1)$$
$$\phi_1 = 53.13°$$
$$v = 500 \sin(100\pi t + 53.13°)$$

(b) $i = I_m \sin(2\pi ft + \phi_2)$

At $t = 0$,

$$4 = 10 \sin(2\pi \times 50 \times 0 + \phi_2)$$
$$\phi_2 = 23.58°$$
$$i = 10 \sin(100\pi t + 23.58°)$$

(ii) Instantaneous values of potential difference and current at $t = 0.015$ second

(a) $v = 500 \sin(100 \times 180 \times 0.015 + 53.13°)$ (angle in degrees)

$= -300$ V

(b) $i = 10 \sin(100 \times 180 \times 0.015 + 23.58°)$ (angle in degrees)

$= -9.17$ A

(iii) Phase angle between potential difference and current

$$\phi = \phi_1 - \phi_2 = 53.13° - 23.58° = 29.55°$$

Example 17

Find the following parameters of a voltage v = 200 sin 314 t:
(i) frequency, (ii) form factor, and (iii) crest factor.

Solution $v = 200 \sin 314\, t$

(i) Frequency

$$v = V_m \sin 2\pi ft$$
$$2\pi f = 314$$
$$f = \frac{314}{2\pi} = 50 \text{ Hz}$$

For a sinusoidal waveform,

$$V_{avg} = \frac{2V_m}{\pi}$$
$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

(ii) Form factor

$$k_f = \frac{V_{rms}}{V_{avg}} = \frac{\frac{V_m}{\sqrt{2}}}{\frac{2V_m}{\pi}} = 1.11$$

(iii) Crest factor

$$k_p = \frac{V_m}{V_{\text{rms}}} = \frac{V_m}{\frac{V_m}{\sqrt{2}}} = 1.414$$

Example 18

A non-sinusoidal voltage has a form factor of 1.2 and peak factor of 1.5. If the average value of the voltage is 10 V, calculate (i) rms value, and (ii) maximum value.

Solution

$$k_f = 1.2$$
$$k_p = 1.5$$
$$V_{\text{avg}} = 10$$

(i) rms value

$$k_f = \frac{V_{\text{rms}}}{V_{\text{avg}}}$$
$$1.2 = \frac{V_{\text{rms}}}{10}$$
$$V_{\text{rms}} = 12 \text{ V}$$

(ii) Maximum value

$$k_p = \frac{V_m}{V_{\text{rms}}}$$
$$1.5 = \frac{V_m}{12}$$
$$V_m = 18 \text{ V}$$

Example 19

The waveform of a voltage has a form factor of 1.15 and a peak factor of 1.5. If the maximum value of the voltage is 4500 V, calculate the average value and rms value of the voltage.

Solution

$$k_f = 1.15$$
$$k_p = 1.5$$
$$V_m = 4500 \text{ V}$$

(i) rms value of the voltage

$$k_p = \frac{V_m}{V_{\text{rms}}}$$
$$1.5 = \frac{4500}{V_{\text{rms}}}$$
$$V_{\text{rms}} = 3000 \text{ V}$$

(ii) Average value of the voltage

$$k_f = \frac{V_{\text{rms}}}{V_{\text{avg}}}$$

$$1.15 = \frac{3000}{V_{\text{avg}}}$$

$$V_{\text{avg}} = 2608.7 \text{ V}$$

Example 20

A 50 Hz sinusoidal current has a peak factor of 1.4 and a form factor of 1.1. Its average value is 20 A. The instantaneous value of the current is 15 A at t = 0. Write the equation of the current and draw its waveform.

Solution

$$f = 50 \text{ Hz}$$
$$k_p = 1.4$$
$$k_f = 1.1$$
$$I_{\text{avg}} = 20 \text{ A}$$
$$i(0) = 15 \text{ A}$$

(i) Equation of current

$$k_f = \frac{I_{\text{rms}}}{I_{\text{avg}}}$$

$$1.1 = \frac{I_{\text{rms}}}{20}$$

$$I_{\text{rms}} = 22 \text{ A}$$

$$k_p = \frac{I_m}{I_{\text{rms}}}$$

$$1.4 = \frac{I_m}{22}$$

$$I_m = 30.8 \text{ A}$$

Since $i = 15$ A at $t = 0$, the current is leading in nature.

$$i = I_m \sin(2\pi f t + \phi)$$

At $t = 0$,

$$15 = 30.8 \sin(2\pi \times 50 \times 0 + \phi)$$
$$\phi = 29.14°$$
$$i = 30.8 \sin(100\pi t + 29.14°)$$

(ii) The waveform is shown is Fig. 3.17.

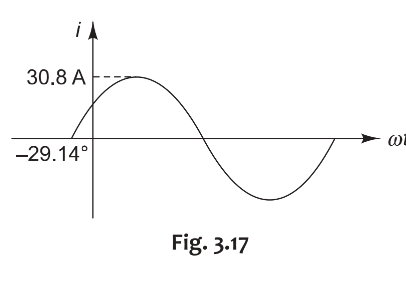


Fig. 3.17

Example 21

Find the average value and rms value of the waveform shown in Fig. 3.18.

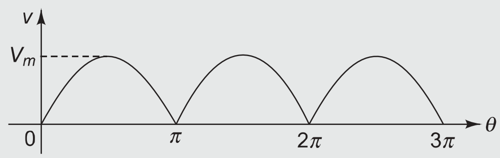


Fig. 3.18

Solution $v = V_m \sin\theta \qquad 0 < \theta < \pi$

(i) Average value of the waveform

$$V_{avg} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\, d\theta$$

$$= \frac{V_m}{\pi}[-\cos\theta]_0^{\pi}$$

$$= \frac{V_m}{\pi}[1+1]$$

$$= \frac{2V_m}{\pi}$$

$$= 0.637\, V_m$$

(ii) rms value of the waveform

$$V_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2(\theta)\, d\theta}$$

$$= \sqrt{\frac{1}{\pi}\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi} \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta}$$

$$= \sqrt{\frac{V_m^2}{\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^\pi}$$

$$= \sqrt{\frac{V_m^2}{\pi}\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - 0 + \frac{\sin 0}{4}\right]}$$

$$= \sqrt{\frac{V_m^2}{2}}$$

$$= \frac{V_m}{\sqrt{2}}$$

$$= 0.707\ V_m$$

Example 22

Find the average and rms values of the waveform shown in Fig. 3.19.

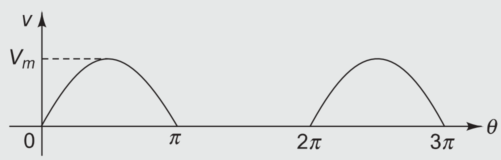


Fig. 3.19

Solution $\quad v = V_m \sin\theta \qquad 0 < \theta < \pi$

$\quad = 0 \qquad \pi < \theta < 2\pi$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} V_m \sin\theta\ d\theta + \int_\pi^{2\pi} 0\,d\theta\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} V_m \sin\theta\ d\theta$$

$$= \frac{V_m}{2\pi}[-\cos\theta]_0^\pi$$

$$= \frac{V_m}{2\pi}[1+1]$$

$$= \frac{V_m}{\pi}$$

$$= 0.318\ V_m$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\left[\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta + \int_{\pi}^{2\pi} 0\, d\theta\right]}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{\pi} V_m^2 \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi} \sin^2\theta\, d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - 0 + \frac{\sin 0}{4}\right]}$$

$$= \sqrt{\frac{V_m^2}{4}}$$

$$= \frac{V_m}{2}$$

$$= 0.5\, V_m$$

Example 23

Find the average value and rms value of the waveform shown in Fig. 3.20.

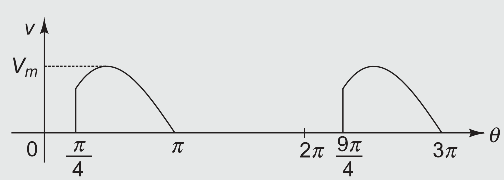


Fig. 3.20

Solution

$$\begin{aligned} v &= 0 && 0 < \theta < \pi/4 \\ &= V_m \sin\theta && \pi/4 < \theta < \pi \\ &= 0 && \pi < \theta < 2\pi \end{aligned}$$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_{\pi/4}^{\pi} V_m \sin\theta\, d\theta$$

$$= \frac{V_m}{2\pi}[-\cos\theta]_{\pi/4}^{\pi}$$

$$= \frac{V_m}{2\pi}\ [1 + 0.707]$$

$$= 0.272\ V_m$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2(\theta)\ d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_{\pi/4}^{\pi} V_m^2 \sin^2\theta\ d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_{\pi/4}^{\pi} \sin^2\theta\ d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\int_{\pi/4}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/4}^{\pi}}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - \frac{\pi}{8} + \frac{\sin \pi/2}{4}\right]}$$

$$= \sqrt{0.227 V_m^2}$$

$$= 0.476\ V_m$$

Example 24

A full-wave rectified wave is clipped at 70.7% of its maximum value as shown in Fig. 3.21. Find its average and rms values.

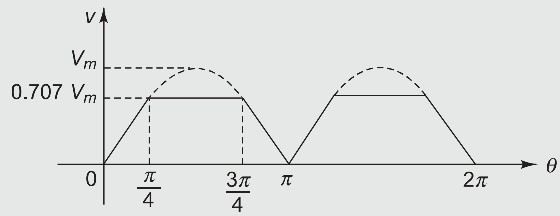


Fig. 3.21

Solution

$$\begin{aligned} v &= V_m \sin\theta && 0 < \theta < \pi/4 \\ &= 0.707\, V_m && \pi/4 < \theta < 3\pi/4 \\ &= V_m \sin\theta && 3\pi/4 < \theta < \pi \end{aligned}$$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{\pi}\int_0^{\pi} v(\theta)\, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/4} V_m \sin\theta\ d\theta + \int_{\pi/4}^{3\pi/4} 0.707 V_m\, d\theta + \int_{3\pi/4}^{\pi} V_m \sin\theta\ d\theta\right]$$

$$= \frac{V_m}{\pi}\left\{\left[-\cos\theta\right]_0^{\pi/4} + 0.707[\theta]_{\pi/4}^{3\pi/4} + \left[-\cos\theta\right]_{3\pi/4}^{\pi}\right\}$$

$$= \frac{V_m}{\pi}(0.293 + 1.11 + 0.293)$$

$$= 0.54\ V_m$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2(\theta)\, d\theta}$$

$$= \sqrt{\frac{1}{\pi}\left[\int_0^{\pi/4} V_m^2 \sin^2\theta\ d\theta + \int_{\pi/4}^{3\pi/4} (0.707 V_m)^2\, d\theta + \int_{3\pi/4}^{\pi} V_m^2 \sin^2\theta\ d\theta\right]}$$

$$= \sqrt{\frac{V_m^2}{\pi}\left\{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/4} + 0.499[\theta]_{\pi/4}^{3\pi/4} + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{3\pi/4}^{\pi}\right\}}$$

$$= \sqrt{0.341\,V_m^2}$$
$$= 0.584\ V_m$$

Example 25

Find the rms value for the given waveform as shown in Fig. 3.22.

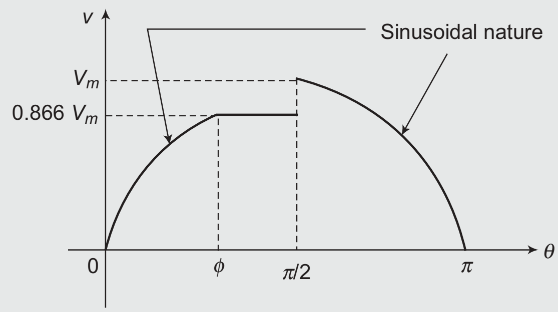


Fig. 3.22 **[Dec 2014]**

Solution

At $\theta = \phi$, $\quad v = 0.866\ V_m$

$$v = V_m \sin\theta$$
$$0.866\ V_m = V_m \sin\phi$$
$$\phi = \frac{\pi}{3}$$

$$v = V_m \sin\theta \qquad 0 < \theta < \frac{\pi}{3}$$
$$= 0.866\ V_m \qquad \frac{\pi}{3} < \theta < \frac{\pi}{2}$$
$$= V_m \sin\theta \qquad \frac{\pi}{2} < \theta < \pi$$

$$V_{\text{rms}} = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2(\theta)\,\mathrm{d}\theta}$$
$$= \sqrt{\frac{1}{\pi}\left[\int_0^{\pi/3} V_m^2 \sin^2\theta\,\mathrm{d}\theta + \int_{\pi/3}^{\pi/2} (0.866\,V_m)^2\,\mathrm{d}\theta + \int_{\pi/2}^{\pi} V_m^2 \sin^2\theta\,\mathrm{d}\theta\right]}$$
$$= \sqrt{\frac{V_m^2}{\pi}\left\{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/3} + 0.75\left[\theta\right]_{\pi/3}^{\pi/2} + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}\right\}}$$
$$= \sqrt{0.472\,V_m^2}$$
$$= 0.687\ V_m$$

Example 26

Find the rms value of the waveform shown in Fig. 3.23.

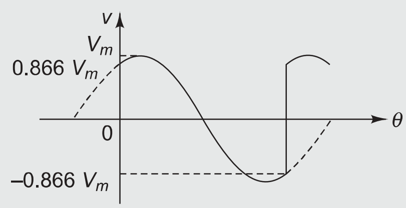


Fig. 3.23

Solution The equation of the waveform is given by $v = V_m \sin(\theta + \phi)$ where ϕ is the phase difference.

When $\theta = 0$, $v = 0.866\ V_m$.

$$0.866\ V_m = V_m \sin(0 + \phi)$$

$$\phi = \sin^{-1}(0.866) = \frac{\pi}{3}$$

$$v = V_m \sin\left(\theta + \frac{\pi}{3}\right)$$

The time period of a complete sine wave is always 2π. Since some part of the waveform is chopped from both the sides,

$$\text{Time period} = 2\pi - \frac{\pi}{3} - \frac{\pi}{3} = \frac{4\pi}{3}$$

$$V_{\text{rms}} = \sqrt{\frac{1}{4\pi/3}\int_0^{4\pi/3} V_m^2 \sin^2\left(\theta + \frac{\pi}{3}\right) d\theta}$$

$$= \sqrt{\frac{3}{4\pi}\int_0^{4\pi/3} V_m^2 \sin^2\left(\theta + \frac{\pi}{3}\right) d\theta}$$

$$= \sqrt{\frac{3V_m^2}{4\pi}\int_0^{4\pi/3}\left[\frac{1 - \cos 2(\theta + \pi/3)}{2}\right] d\theta}$$

$$= \sqrt{\frac{3V_m^2}{4\pi}\left[\frac{\theta}{2} - \frac{\sin 2(\theta + \pi/3)}{4}\right]_0^{4\pi/3}}$$

$$= \sqrt{0.6031\, V_m^2}$$

$$= 0.776\ V_m$$

Example 27

Find the average and rms values of the waveform shown in Fig. 3.24.

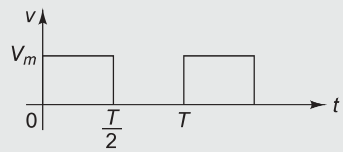


Fig. 3.24

Solution

$$v = V_m \qquad 0 < t < T/2$$
$$= 0 \qquad T/2 < t < T$$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt$$
$$= \frac{1}{T}\left[\int_0^{T/2} V_m\,dt + \int_{T/2}^{T} 0\,dt\right]$$
$$= \frac{1}{T}\int_0^{T/2} V_m\,dt$$
$$= \frac{V_m}{T}[t]_0^{T/2}$$
$$= \frac{V_m}{T}\cdot\frac{T}{2}$$
$$= 0.5\ V_m$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$
$$= \sqrt{\frac{1}{T}\int_0^{T/2} V_m^2\,dt}$$
$$= \sqrt{\frac{V_m^2}{T}[t]_0^{T/2}}$$
$$= \sqrt{\frac{V_m^2}{T}\cdot\frac{T}{2}}$$

$$= \sqrt{\frac{V_m^2}{2}}$$

$$= 0.707\ V_m$$

Example 28

Find the average value of the waveform shown in Fig. 3.25.

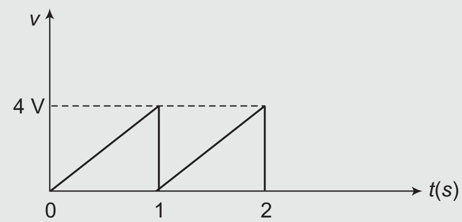


Fig. 3.25 **[Dec 2014]**

Solution

$$v = 4t$$

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,\mathrm{d}t$$

$$= \frac{1}{1}\int_0^1 4t\ \mathrm{d}t$$

$$= 4\left[\frac{t^2}{2}\right]_0^1$$

$$= 4\left(\frac{1}{2} - 0\right)$$

$$= 2\ \text{V}$$

Example 29

Find the rms value for the given waveform in Fig. 3.26.

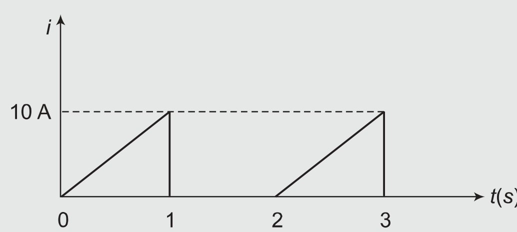


Fig. 3.26 **[May 2015]**

Solution

$$i = 10\,t \qquad 0 < t < 1$$
$$= 0 \qquad 1 < t < 2$$

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,\mathrm{d}t}$$
$$= \sqrt{\frac{1}{2}\left[\int_0^1 (10t)^2\,\mathrm{d}t + \int_1^2 0\,\mathrm{d}t\right]}$$
$$= \sqrt{\frac{1}{2}\times 100\left[\frac{t^3}{3}\right]_0^1}$$
$$= \sqrt{\frac{100}{2}\left[\frac{1}{3}-0\right]}$$
$$= 4.084$$

Example 30

Determine the rms value of the voltage waveform shown in Fig. 3.27.

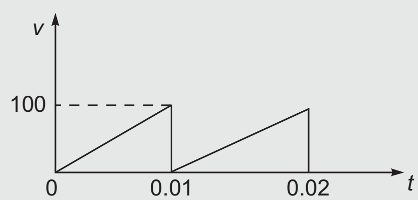


Fig. 3.27

[May 2013]

Solution

$$v(t) = \frac{100}{0.01}\,t = 10000\,t \qquad 0 < t < 0.01$$

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$
$$= \sqrt{\frac{1}{0.01}\int_0^{0.01} (10000t)^2 dt}$$
$$= \sqrt{10^{10}\left[\frac{t^3}{3}\right]_0^{0.01}}$$

$$= \sqrt{10^{10}\left[\frac{(0.01)^3}{3} - 0\right]}$$

$$= 57.74 \text{ V}$$

Example 31

Find the average and rms values of the waveform shown in Fig. 3.28.

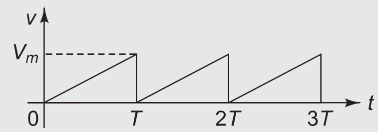


Fig. 3.28

Solution $\quad v = \dfrac{V_m}{T} t \qquad 0 < t < T$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{V_m}{T} t \ dt$$

$$= \frac{V_m}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{V_m}{T^2} \cdot \frac{T^2}{2}$$

$$= 0.5 \ V_m$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T \frac{V_m^2}{T^2} .t^2 \ dt}$$

$$= \sqrt{\frac{V_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^T}$$

$$= \sqrt{\frac{V_m^2}{T^3}\left[\frac{T^3}{3}\right]}$$

$$= \sqrt{\frac{V_m^2}{3}}$$

$$= 0.577\ V_m$$

Example 32

Find the average and rms values of the waveform shown in Fig. 3.29.

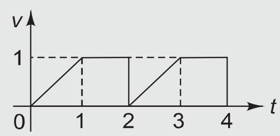


Fig. 3.29 [May 2016]

Solution

$$v = t \qquad 0 < t < 1$$
$$= 1 \qquad 1 < t < 2$$

(i) Average value of the waveform

$$V_{\text{avg}} = \frac{1}{T}\int_0^T v(t)\,dt$$

$$= \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 1\,dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{t^2}{2}\right]_0^1 + [t]_1^2\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2} - 0 + 2 - 1\right]$$

$$= 0.75 \text{ V}$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$= \sqrt{\frac{1}{2}\left[\int_0^1 t^2 dt + \int_1^2 (1)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{2}\left\{\left[\frac{t^3}{3}\right]_0^1 + [t]_1^2\right\}}$$

$$= \sqrt{\frac{1}{2}\left[\frac{1}{3} - 0 + 2 - 1\right]}$$

$$= 0.816 \text{ V}$$

Example 33

Find the average and rms values of the waveform shown in Fig. 3.30.

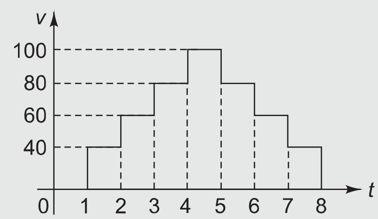


Fig. 3.30

Solution

(i) Average value of the waveform

$$V_{avg} = \frac{0+40+60+80+100+80+60+40}{8} = 57.5 \text{ V}$$

(ii) rms value of the waveform

$$V_{rms} = \sqrt{\frac{0^2+(40)^2+(60)^2+(80)^2+(100)^2+(80)^2+(60)^2+(40)^2}{8}} = 64.42 \text{ V}$$

Example 34

Find the average and rms values of the waveform shown in Fig. 3.31.

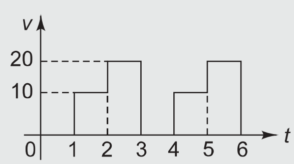


Fig. 3.31

Solution

(i) Average value of the waveform

$$V_{avg} = \frac{0+10+20}{3} = 10 \text{ V}$$

(ii) rms value of the waveform

$$V_{\text{rms}} = \sqrt{\frac{0^2 + (10)^2 + (20)^2}{3}} = 12.9 \text{ V}$$

Example 35

Find the effective value of the resultant current which carries simultaneously a direct current of 10 A and a sinusoidally alternating current with a peak value of 10 A.

Solution

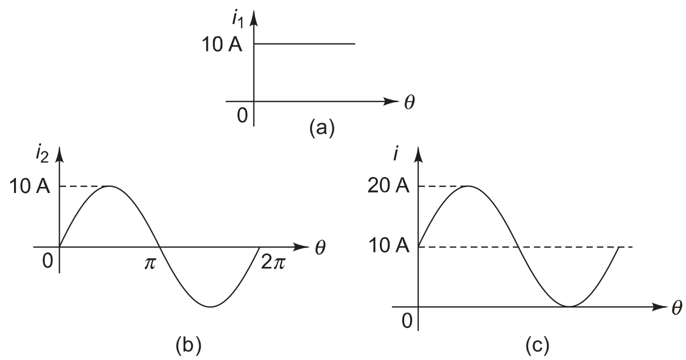


Fig. 3.32

$$i = i_1 + i_2 = 10 + 10 \sin \theta$$

$$I_{\text{eff}} = I_{\text{rms}} = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} i^2(\theta)\, d\theta}$$

$$= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (10 + 10 \sin \theta)^2\, d\theta}$$

$$= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (100 + 200 \sin \theta + 100 \sin^2 \theta)\, d\theta}$$

$$= \sqrt{\frac{100}{2\pi} \int_0^{2\pi} (1 + 2 \sin \theta + \sin^2 \theta)\, d\theta}$$

$$= \sqrt{\frac{100}{2\pi} \int_0^{2\pi} \left[1 + 2 \sin \theta + \left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta}$$

$$= \sqrt{\frac{100}{2\pi}\left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}}$$

$$= \sqrt{\frac{100}{2\pi}\left[2\pi - 2\cos 2\pi + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} - 0 + 2\cos 0 - 0 + \frac{\sin 0}{4}\right]}$$

$$= \sqrt{\frac{100}{2\pi}\left[2\pi - 2 + \frac{2\pi}{2} + 2\right]}$$

$$= \sqrt{\frac{100}{2\pi} \times 3\pi}$$

$$= 12.25 \text{ A}$$

Example 36

Find the effective value of a resultant current in a wire which carries simultaneously a direct current of i_1 = 10 A and alternating current given by $i_2 = 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$.

Solution

$$i_1 = 10 \text{ A}$$

$$i_2 = 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$$

$$i = i_1 + i_2$$

$$= 10 + 12 \sin \omega t + 6 \sin\left(3\omega t - \frac{\pi}{6}\right) + 4 \sin\left(5\omega t + \frac{\pi}{3}\right)$$

$$= 10 + 12 \sin \theta + 6 \sin (3\theta - 30°) + 4 \sin (5\theta + 60°)$$

$$I_{\text{eff}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} i^2(\theta)\,d\theta}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} [10 + 12\sin\theta + 6\sin(3\theta - 30°) + 4\sin(5\theta + 60°)]^2\,d\theta}$$

$$= 14.07 \text{ A}$$

Example 37

Find the relative heating effects of two current waves of equal peak value, one sinusoidal and the other, rectangular in shape.

Solution

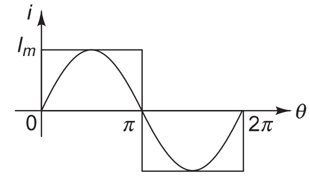


Fig. 3.33

$$\text{rms value of the rectangular wave} = I_m$$

$$\text{rms value of the sinusoidal current wave} = \frac{I_m}{\sqrt{2}}$$

$$\text{Heating effect due to the rectangular current wave} = (I_m)^2\, RT$$

$$\text{Heating effect due to the sinusoidal current wave} = \left(\frac{I_m}{\sqrt{2}}\right)^2 RT = \frac{(I_m)^2}{2}\; RT$$

$$\text{Relative heating effects} = \frac{(I_m)^2}{2} RT : (I_m)^2\, RT$$

$$= \frac{1}{2} : 1$$

$$= 1 : 2$$

Useful Formulae

Average Value and rms value

$$F_{\text{avg}} = \frac{1}{T}\int_0^T f(t)dt \qquad k_P = \frac{\text{max. value}}{\text{rms value}}$$

$$F_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T f^2(t)dt} \qquad k_f = \frac{\text{rms value}}{\text{avg. value}}$$

Exercise 3.1

3.1 An alternating current varying sinusoidally with a frequency of 50 Hz has an rms value of 20 A. Write down the equation for the instantaneous value and find this value (i) 0.0025 second, and (ii) 0.0125 second after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A? *[$i = 28.28 \sin 100\,\pi\, t$, 20 A, –20 A, $\frac{1}{300}$ s]*

Determine the value of the capacitance of the capacitor.

[10.42 ∠– 38.56° A, 6.61 ∠34.21° A, 13.95 ∠– 11.56° A, 0.98 (lagging) 54.9 μF]

4.27 A resistor of 30 Ω and a capacitor of unknown value are connected in parallel across a 110 V, 50 Hz supply. The combination draws a current of 5 A form the supply. Find the value of the unknown capacitance of the capacitor. This combination is again connected across a 110 V supply of unknown frequency. It is observed that the total current drawn from the mains falls to 4 A. Determine the frequency of the supply. *[98.58 μF, 23.68 Hz]*

4.28 Two reactive circuits have an impedance of 20 Ω each. One of them has a lagging power factor of 0.8 and the other has a leading power factor of 0.6. Find (i) voltage necessary to send a current of 10 A through the two in series, and (ii) current drawn from 200 V supply if the two are connected in parallel. Draw a phasor diagram in each case. *[282.8 V, 14.14 A]*

4.29 Inductor loads of 0.8 kW and 1.2 kW at lagging power factors of 0.8 and 0.6 respectively are connected across a 200 V, 50 Hz supply. Find the total current, power factor and the value of the capacitor to be put in parallel to both to raise the overall power factor to 0.9 lagging. *[14.87 A, 0.673 (lagging), 98 μF]*

4.8 SERIES RESONANCE

[Dec 2013, May 2016]

A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it. At resonance, the circuit thus behaves as a pure resistor and the net reactance is zero.

Consider the series *R-L-C* circuit as shown in Fig. 4.93. The impedance of the circuit is

$$\begin{aligned}\bar{Z} &= R + jX_L - jX_C \\ &= R + j\omega L - j\frac{1}{\omega C} \\ &= R + j\left(\omega L - \frac{1}{\omega C}\right)\end{aligned}$$

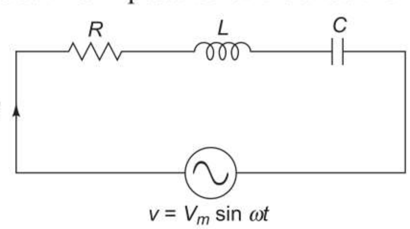


Fig. 4.93 *Series circuit*

At resonance, Z must be resistive. Therefore, the condition for resonance is

$$\begin{aligned}\omega L - \frac{1}{\omega C} &= 0 \\ \omega = \omega_0 &= \frac{1}{\sqrt{LC}} \\ f = f_0 &= \frac{1}{2\pi\sqrt{LC}}\end{aligned}$$

where f_0 is called the resonant frequency of the circuit.

Power Factor

$$\text{Power factor} = \cos\phi = \frac{R}{Z}$$

At resonance $\quad Z = R$

$$\text{Power factor} = \frac{R}{R} = 1$$

Current Since impedance is minimum, the current is maximum at resonance. Thus, the circuit accepts more current and as such, an R-L-C circuit under resonance is called an *acceptor circuit*.

$$I_0 = \frac{V}{Z} = \frac{V}{R}$$

Voltage At resonance,

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 L\, I_0 = \frac{1}{\omega_0 C} I_0$$

$$V_{L_0} = V_{C_0}$$

Thus, potential difference across inductor equal to potential difference across capacitor being equal and opposite cancel each other. Also, since I_0 is maximum, V_{L_0} and V_{C_0} will also be maximum. Thus, voltage magnification takes place during resonance. Hence, it is also referred to as voltage magnification circuit.

Phasor Diagram The phasor diagram is shown in Fig. 4.94.

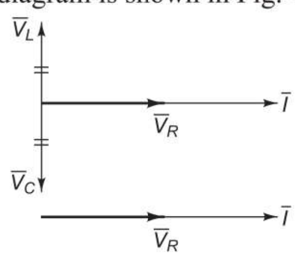


Fig. 4.94 *Phasor diagram*

Behaviour of R, L and C with Change in Frequency Resistance remains constant with the change in frequencies. Inductive reactance X_L is directly proportional to frequency f. It can be drawn as a straight line passing through the origin. Capacitive reactance X_C is inversely proportional to the frequency f. It can be drawn as a rectangular hyperbola in the fourth quadrant.

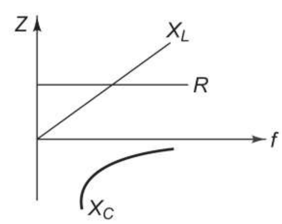


Fig. 4.95 *Behaviour of R, L and C with change in frequency*

Total impedance $\overline{Z} = R + j\,(X_L - X_C)$

(a) When $f < f_0$, impedance is capacitive and decreases up to f_0. The power factor is leading in nature.

(b) At $f = f_0$, impedance is resistive. The power factor is unity.
(c) When $f > f_0$, impedance is inductive and goes on increasing beyond f_0. The power factor is lagging in nature.

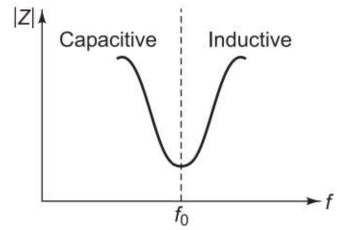


Fig. 4.96 *Impedance*

Bandwidth For the series *R-L-C* circuit, bandwidth is defined as the range of frequencies for which the power delivered to R is greater than or equal to $\frac{P_0}{2}$ where P_0 is the power delivered to R at resonance. From the shape of the resonance curve, it is clear that there are two frequencies for which the power delivered to R is half the power at resonance. For this reason, these frequencies are referred as those corresponding to the half-power points. The magnitude of the current at each half-power point is the same.

Hence,
$$I_1^2 R = \frac{1}{2} I_0^2 R = I_2^2 R$$

where the subscript 1 denotes the lower half point and the subscript 2, the higher half point. It follows then that

$$I_1 = I_2 = \frac{I_0}{\sqrt{2}} = 0.707 I_0$$

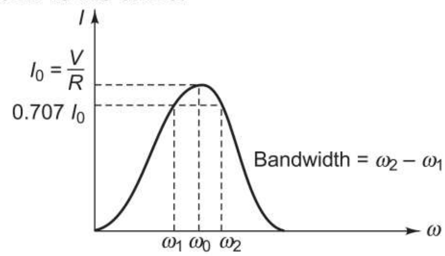


Fig. 4.97 *Resonance curve*

Accordingly, the bandwidth may be identified on the resonance curve as the range of frequencies over which the magnitude of the current is equal to or greater than 0.707 of the current at resonance. In Fig. 4.97, the bandwidth is $\omega_2 - \omega_1$.

Expression for Bandwidth Generally, at any frequency ω,

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{4.1}$$

At half-power points,

$$I = \frac{I_0}{\sqrt{2}}$$

But
$$I_0 = \frac{V}{R}$$

$$I = \frac{V}{\sqrt{2}R} \tag{4.2}$$

From Eqs (4.1) and (4.2),

$$\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V}{\sqrt{2}R}$$

$$\frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{1}{\sqrt{2}R}$$

Squaring both the sides,

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\omega L - \frac{1}{\omega C} \pm R = 0$$

$$\omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$$

$$\omega = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}$$

For low values of R, the term $\left(\frac{R^2}{4L^2}\right)$ can be neglected in comparison with the term $\frac{1}{LC}$.

Then ω is given by, $\omega = \pm\frac{R}{2L} \pm \sqrt{\frac{1}{LC}} = \pm\frac{R}{2L} \pm \frac{1}{\sqrt{LC}}$

The resonant frequency for this circuit is given by

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\omega = \pm\frac{R}{2L} + \omega_0 \qquad \text{(considering only positive sign of } \omega_0\text{)}$$

$$\omega_1 = \omega_0 - \frac{R}{2L}$$

and $$\omega_2 = \omega_0 + \frac{R}{2L}$$

$$f_1 = f_0 - \frac{R}{4\pi L}$$

and $$f_2 = f_0 + \frac{R}{4\pi L}$$

$$\text{Bandwidth} = \omega_2 - \omega_1 = \frac{R}{L}$$

or $$\text{Bandwidth} = f_2 - f_1 = \frac{R}{2\pi L}$$

Quality Factor It is a measure of voltage magnification in the series resonant circuit. It is also a measure of selectivity or sharpness of the series resonant circuit.

$$Q_0 = \frac{\text{Voltage across inductor or capacitor}}{\text{Voltage at resonance}}$$

$$= \frac{V_{L_0}}{V} = \frac{V_{C_0}}{V}$$

Substituting values of V_{L_0} and V,

$$Q_0 = \frac{I_0 X_{L_0}}{I_0 R}$$

$$= \frac{X_{L_0}}{R}$$

$$= \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Substituting values of ω_0,

$$Q_0 = \frac{\left(\frac{1}{\sqrt{LC}}\right)L}{R}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}}$$

Example 1

A series R-L-C circuit has the following parameter values: $R = 10\ \Omega$, $L = 0.01$ H, $C = 100\ \mu$F. Compute the resonant frequency, bandwidth, and lower and upper frequencies of the bandwidth.

Solution $R = 10\ \Omega$

$L = 0.01$ H

$C = 100\ \mu$F

(i) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01\times 100\times 10^{-6}}} = 159.15\text{ Hz}$$

(ii) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{10}{2\pi\times 0.01} = 159.15\text{ Hz}$$

(iii) Lower frequency of bandwidth

$$f_1 = f_0 - \frac{BW}{2} = 159.15 - \frac{159.15}{2} = 79.58\text{ Hz}$$

(iv) Upper frequency of bandwidth

$$f_2 = f_0 + \frac{BW}{2} = 159.15 + \frac{159.15}{2} = 238.73\text{ Hz}$$

Example 2

For a series RLC circuit having R = 10 Ω, L = 0.01 H and C = 100 μF, Find the resonant frequency, quality factor and bandwith. **[Dec 2014]**

Solution $R = 10\ \Omega$

$L = 0.01$ H

$C = 100\ \mu$F

(i) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01\times 100\times 10^{-6}}} = 159.15\text{ Hz}$$

(ii) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.01}{100\times 10^{-6}}} = 1$$

(iii) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{10}{2\pi\times 0.01} = 159.15\text{ Hz}$$

Example 3

A series RLC circuit has the following parameter values: R = 10 Ω, L = 0.014 H, C = 100 μF. Compute the resonant frequency, quality factor, bandwidth, lower cut-off frequency and upper cut-off frequency. **[May 2015]**

Solution $R = 10\ \Omega$

$L = 0.014$ H

$C = 100\ \mu$F

(i) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.014 \times 100 \times 10^{-6}}} = 134.51\ \text{kHz}$$

(ii) Qualify factor

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.014}{100 \times 10^{-6}}} = 1.18$$

(iii) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{10}{2\pi \times 0.014} = 113.68\ \text{Hz}$$

(iv) Lower cut-off frequency (f_1)

$$f_1 = f_0 - \frac{BW}{2} = 134.51 - \frac{113.68}{2} = 77.67\ \text{Hz}$$

(v) Upper cut-off frequency (f_2)

$$f_2 = f_0 + \frac{BW}{2} = 134.51 + \frac{113.68}{2} = 191.35\ \text{Hz}$$

Example 4

A series R-L-C circuit consists of R = 1000 Ω, L = 100 mH and C = 10 μF. The applied voltage across the circuit is 100 V.

(i) Find the resonance frequency of the circuit.

(ii) Find Q of the circuit at resonant frequency.

(iii) Calculate the bandwidth of the circuit. **[May 2016]**

Solution $R = 1000\ \Omega$

$L = 100$ mH

$C = 10\ \mu$F

$V = 100$ V

(i) Resonance frequency of the circuit

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{100 \times 10^{-3} \times 10 \times 10^{-6}}} = 159.15\ \text{kHz}$$

(ii) Q of the circuit at resonant frequency

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{1000}\sqrt{\frac{100 \times 10^{-3}}{10 \times 10^{-6}}} = 0.1$$

(iii) Bandwidth

$$BW = \frac{R}{2\pi L} = \frac{1000}{2\pi \times 100 \times 10^{-3}} = 1591.55 \text{ Hz}$$

Example 5

An R-L-C series circuit with a resistance of 10 Ω, inductance of 0.2 H and a capacitance of 40 μF is supplied with a 100 V supply at variable frequency. Find the following w.r.t. the series resonant circuit:

(i) frequency at which resonance takes place
(ii) current
(iii) power
(iv) power factor
(v) voltage across R-L-C at that time
(vi) quality factor
(vii) half-power points
(viii) resonance and phasor diagrams

Solution

$$R = 10\ \Omega$$
$$L = 0.2 \text{ H}$$
$$C = 40\ \mu\text{F}$$
$$V = 100 \text{ V}$$

(i) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2 \times 40 \times 10^{-6}}} = 56.3 \text{ Hz}$$

(ii) Current

$$I_0 = \frac{V}{R} = \frac{100}{10} = 10 \text{ A}$$

(iii) Power

$$P_0 = I_0^2 R = (10)^2 \times 10 = 1000 \text{ W}$$

(iv) Power factor

$$\text{pf} = 1$$

(v) Voltage across R, L, C

$$V_{R_0} = R\, I_0 = 10 \times 10 = 100 \text{ V}$$
$$V_{L_0} = X_{L_0} I_0 = 2\pi f_0 L I_0 = 2\pi \times 56.3 \times 0.2 \times 10 = 707.5 \text{ V}$$
$$V_{C_0} = X_{C_0} I_0 = \frac{1}{2\pi f_0 C} I_0 = \frac{1}{2\pi \times 56.3 \times 40 \times 10^{-6}} \times 10 = 707.5 \text{ V}$$

(vi) Quality factor

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.2}{40 \times 10^{-6}}} = 7.07$$

(vii) Half-power points

$$f_1 = f_0 - \frac{R}{4\pi L} = 56.3 - \frac{10}{4\pi \times 0.2} = 52.32\,\text{Hz}$$

$$f_2 = f_0 + \frac{R}{4\pi L} = 56.3 + \frac{10}{4\pi \times 0.2} = 60.3\,\text{Hz}$$

(viii) Resonance and phasor diagram

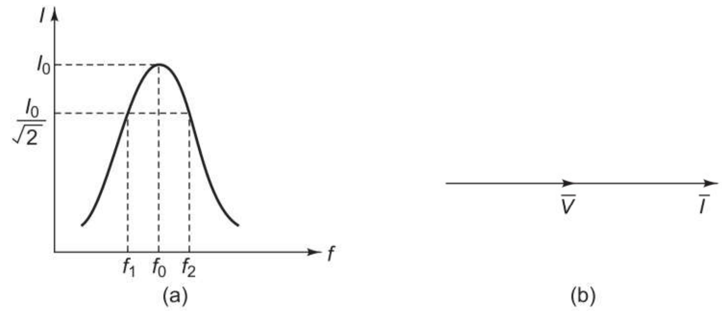


Fig. 4.98

Example 6

A series R-L-C circuit is supplied with v(t) = 10 sin 1000 t volts. If the maximum peak voltage across capacitor is 400 volts, find the quality factor of the circuit. **[Dec 2015]**

Solution

$$V_m = 10$$

$$V_{C_m} = 400\text{ V}$$

$$V = \frac{V_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}\text{ V}$$

$$V_{co} = \frac{V_{cm}}{\sqrt{2}} = \frac{400}{\sqrt{2}} = 200\sqrt{2}\text{ V}$$

$$Q = \frac{V_{co}}{V} = \frac{200\sqrt{2}}{5\sqrt{2}} = 40$$

Example 7

A series R-L-C circuit is connected to a 200 V ac supply. The current drawn by the circuit at resonance is 20 A. The voltage drop across the capacitor is 5000 V at series resonance. Calculate resistance and inductance if capacitance is 4 μF. Also, calculate the resonant frequency.

Solution

$$V = 200\text{ V}$$

$$I_0 = 20\text{ A}$$

$$V_{C_0} = 5000\text{ V}$$

$$C = 4\ \mu\text{F}$$

(i) Resistance

$$R = \frac{V}{I_0} = \frac{200}{20} = 10\ \Omega$$

(ii) Resonant frequency

$$X_{C_0} = \frac{V_{C_0}}{I_0} = \frac{5000}{20} = 250\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$250 = \frac{1}{2\pi \times f_0 \times 4 \times 10^{-6}}$$

$$f_0 = 159.15\ \text{Hz}$$

(iii) Inductance

At resonance $X_{C_0} = X_{L_0} = 250\ \Omega$

$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 159.15 \times L$$

$$L = 0.25\ \text{H}$$

Example 8

A resistor and a capacitor are connected in series with a variable inductor. When the circuit is connected to a 230 V, 50 Hz supply, the maximum current obtained by varying the inductance is 2 A. The voltage across the capacitor is 500 V. Calculate the resistance, inductance and capacitance of the circuit. **[Dec 2012]**

Solution

$$V = 230\ \text{V}$$

$$f_0 = 50\ \text{Hz}$$

$$I_0 = 2\ \text{A}$$

$$V_{C_0} = 500\ \text{V}$$

(i) Resistance

$$R = \frac{V}{I_0} = \frac{230}{2} = 115\ \Omega$$

(ii) Capacitance

$$X_{C_0} = \frac{V_{C0}}{I_0} = \frac{500}{2} = 250\ \Omega$$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$250 = \frac{1}{2\pi \times 50 \times C}$$

$$C = 12.73\ \mu\text{F}$$

(iii) Inductance

At resonance $\quad X_{C_0} = X_{L_0} = 250\ \Omega$

$$X_{L_0} = 2\pi f_0 L$$

$$250 = 2\pi \times 50 \times L$$

$$L = 0.795\ \text{H}$$

Example 9

A coil of 2 Ω resistance and 0.01 H inductance is connected in series with a capacitor across 200 V mains. What must be the capacitance in order that maximum current occurs at a frequency of 50 Hz? Find also the current and voltage across the capacitor.

Solution $\quad R = 2\ \Omega$

$$L = 0.01\ \text{H}$$

$$V = 200\ \text{V}$$

$$f_0 = 50\ \text{Hz}$$

(i) Capacitance

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$50 = \frac{1}{2\pi\sqrt{0.01 \times C}}$$

$$C = 1013.2\ \mu\text{F}$$

(ii) Current

$$I_0 = \frac{V}{R} = \frac{200}{2} = 100\ \text{A}$$

(iii) Voltage across capacitor

$$V_{C_0} = I_0\, X_{C_0} = I_0 \frac{1}{2\pi f_0 C} = 100 \times \frac{1}{2\pi \times 50 \times 1013.2 \times 10^{-6}} = 314.16\ \text{V}$$

Example 10

A voltage v(t) = 10 sin ωt is applied to a series R-L-C circuit. At the resonant frequency of the circuit, the voltage across the capacitor is found to be 500 V. The bandwidth of the circuit is known to be 400 rad/s and the impedance of the circuit at resonance is 100 Ω. Determine inductance and capacitance resonant frequency, upper and lower cut-off frequencies.

Solution $\quad v(t) = 10 \sin \omega t$

$$V_{C_0} = 500 \text{ V}$$
$$BW = 400 \text{ rad/s}$$
$$R = 100\ \Omega$$

(i) Inductance and capacitance

$$V = \frac{V_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.07 \text{ V}$$
$$I_0 = \frac{V}{R} = \frac{7.07}{100} = 0.0707 \text{ A}$$
$$BW = \frac{R}{L}$$
$$400 = \frac{100}{L}$$
$$L = 0.25 \text{ H}$$
$$Q_0 = \frac{V_{C0}}{V} = \frac{500}{7.07} = 70.72$$
$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$$
$$70.72 = \frac{1}{100}\sqrt{\frac{0.25}{C}}$$
$$C = 4.99 \text{ nF}$$

(ii) Resonant frequency

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi \times \sqrt{0.25 \times 4.99 \times 10^{-9}}} = 4506.09 \text{ Hz}$$

(iii) Lower cut-off frequency

$$f_1 = f_0 - \frac{R}{4\pi L} = 4506.09 - \frac{100}{4\pi \times 0.25} = 4474.26 \text{ Hz}$$

(iv) Upper cut-off frequency

$$f_2 = f_0 + \frac{R}{4\pi L} = 4506.09 + \frac{100}{4\pi \times 0.25} = 4537.92 \text{ Hz}$$

Example 11

An inductor having a resistance of 25 Ω and Q_0 of 10 at a resonant frequency of 10 kHz is fed from a 100 V supply. Calculate (i) value of series capacitance required to produce resonance with the coil, (ii) the inductance of the coil, (iii) Q_0 using L/C ratio, (iv) voltage across capacitor, and (v) voltage across the coil. **[May 2013]**

Solution

$$R = 25\ \Omega$$
$$Q_0 = 10$$
$$f_0 = 10\ \text{kHz}$$
$$V = 100$$

(i) Value of series capacitance

$$Q_0 = \frac{V_{C0}}{V}$$
$$10 = \frac{V_{C0}}{100}$$
$$V_{C0} = 1000\ \text{V}$$
$$I_0 = \frac{V}{R} = \frac{100}{25} = 4\ \text{A}$$
$$V_{C0} = I_0 X_{C0}$$
$$1000 = 4X_{C0}$$
$$X_{C0} = 250\ \Omega$$
$$X_{C0} = \frac{1}{2\pi f_0 C}$$
$$250 = \frac{1}{2\pi \times 10 \times 10^3 \times C}$$
$$C = 6.37 \times 10^{-8}\ \text{F}$$

(ii) Inductance of the coil

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$10 \times 10^3 = \frac{1}{2\pi\sqrt{L \times 6.37 \times 10^{-8}}}$$
$$L = 3.98\ \text{mH}$$

(iii) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{25}\sqrt{\frac{3.98 \times 10^{-3}}{6.37 \times 10^{-8}}} = 10$$

(iv) Voltage across capacitor

$$V_{L0} = V_{C0} = 1000\ \text{V}$$

(v) Voltage across coil

$$Z_{\text{coil}} = \sqrt{R^2 + X_L^2} = \sqrt{(25)^2 + (250)^2} = 251.25\ \Omega$$

$$V_{\text{coil}} = IZ_{\text{coil}} = 4 \times 251.25 = 1005 \text{ V}$$

Example 12

A series resonant circuit has an impedance of 500 Ω at resonant frequency. Cut-off frequencies are 10 kHz and 100 Hz. Determine (i) resonant frequency, (ii) value of L, C, and (iii) quality factor at resonant frequency.

Solution

$$R = 500\ \Omega$$

$$f_1 = 100 \text{ Hz}$$

$$f_2 = 10 \text{ kHz}$$

(i) Resonant frequency

$$BW = f_2 - f_1 = 10000 - 100 = 9900 \text{ Hz}$$

$$f_1 = f_0 - \frac{R}{4\pi L} \tag{1}$$

$$f_2 = f_0 + \frac{R}{4\pi L} \tag{2}$$

Adding Eqs (1) and (2),

$$f_1 + f_2 = 2f_0$$

$$f_0 = \frac{f_1 + f_2}{2} = \frac{100 + 10000}{2} = 5050 \text{ Hz}$$

(ii) Values of L and C

$$BW = \frac{R}{2\pi L}$$

$$9900 = \frac{500}{2\pi L}$$

$$L = 8.038 \text{ mH}$$

$$X_{L_0} = 2\pi f_0 L = 2\pi \times 5050 \times 8.038 \times 10^{-3} = 255.05\ \Omega$$

At resonance $\quad X_{L_0} = X_{C_0} = 255.05\ \Omega$

$$X_{C_0} = \frac{1}{2\pi f_0 C}$$

$$255.05 = \frac{1}{2\pi \times 5050 \times C}$$

$$C = 0.12\ \mu\text{F}$$

(iii) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{500}\sqrt{\frac{8.038 \times 10^{-3}}{0.12 \times 10^{-6}}} = 0.5176$$

Example 13

Impedance of a circuit is observed to be capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing. Find the values of circuit elements if the power drawn by this circuit is 100 W at 100 Hz, when the current is 1 A. The power factor of the circuit at 70 Hz is 0.707.

Solution

$$f_0 = 100 \text{ Hz}$$
$$P_0 = 100 \text{ W}$$
$$I_0 = 1 \text{ A}$$
$$(\text{pf})_{70 \text{ Hz}} = 0.707$$

The impedance of the circuit is capacitive and decreasing from 1 Hz to 100 Hz. Beyond 100 Hz, the impedance starts increasing.

$$f_0 = 100 \text{ Hz}$$
$$P_0 = I_0^2 R$$
$$100 = (1)^2 \times R$$
$$R = 100 \ \Omega$$
$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
$$100 = \frac{1}{2\pi\sqrt{LC}}$$
$$LC = 2.53 \times 10^{-6} \qquad (1)$$

Power factor at 70 Hz is 0.707.

$$\frac{R}{Z} = 0.707$$
$$\frac{100}{Z} = 0.707$$
$$Z = 141.44 \ \Omega$$

$$\text{Impedance at 70 Hz} = Z_{70} = \sqrt{R^2 + (X_C - X_L)^2}$$
$$141.44 = \sqrt{(100)^2 + \left(\frac{1}{2\pi \times 70 \times C} - 2\pi \times 70 \times L\right)^2}$$
$$\frac{2.27 \times 10^{-3}}{C} - 439.82\, L = 100.02 \qquad (2)$$

Solving Eqs (1) and (2),

$$L = 0.2187 \text{ H}$$
$$C = 11.58 \ \mu\text{F}$$

Example 14

A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor. When the capacitor is set to 500 pF, the current has its maximum value while it is reduced to one-half when the capacitance is 600 pF. Find resistance, inductance and Q-factor of inductor.

Solution

$$f_0 = 1 \text{ MHz}$$
$$C_1 = 500 \text{ pF}$$
$$C_2 = 600 \text{ pF}$$

(i) Resistance and inductance of inductor

At resonance $C = 500$ pF

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$10^6 = \frac{1}{2\pi\sqrt{L \times 500 \times 10^{-12}}}$$

$$L = 0.05 \text{ mH}$$

$$X_L = 2\pi f_0 L = 2\pi \times 10^6 \times 0.05 \times 10^{-3} = 314.16\ \Omega$$

When capacitance is 600 pF, the current reduces to one-half of the current at resonance,

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 10^6 \times 600 \times 10^{-12}} = 265.26\ \Omega$$

$$I = \frac{1}{2} I_0$$

$$\frac{V}{Z} = \frac{1}{2}\frac{V}{R}$$

$$Z = 2R$$

$$\sqrt{R^2 + (X_L - X_C)^2} = 2R$$

$$R^2 + (314.16 - 265.26)^2 = 4R^2$$

$$3R^2 = 2391.21$$

$$R = 28.23\ \Omega$$

(ii) Quality factor

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{28.23}\sqrt{\frac{0.05 \times 10^{-3}}{500 \times 10^{-12}}} = 11.2$$