SOLUTION

(a) In Example 2.1 the value of λ was found as 1.152 Wb-T for $B_c = 1.2$ T. Therefore, for sinusoidai variation of B_c,

$$\lambda = 1.152 \sin 314t \text{ Wb-T}$$

The emf is

$$e = \frac{d\lambda}{dt} = 361.7\cos 314t \quad \text{V}$$

(b)

$$\mathcal{R}_c = \frac{l_c}{\mu_0 \mu_r A_c} = \frac{0.4}{4\pi \times 10^{-7} \times 6000 \times 16 \times 10^{-4}}$$

$$= 3.317 \times 10^4$$

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_g} = \frac{6 \times 10^{-4}}{4\pi \times 10^{-7} \times 16 \times 10^{-4}} = 29.856 \times 10^4$$

(c) From Example 2.1

$$i = 1.06 \text{ A}$$

$$\therefore \quad L = \frac{\lambda}{i} = \frac{1.152}{1.06} = 1.09 \text{ H}$$

It can also be found by using Eq. (2.21). Thus

$$L = N^2 P = \frac{N^2}{\mathcal{R}} = \frac{N^2}{\mathcal{R}_c + \mathcal{R}_g} = \frac{(600)^2}{(3.316 + 29.84) \times 10^4}$$

$$= 1.08 \text{ H}$$

(d) The energy stored in the magnetic field is from Eq. (2.32)

$$W_f = \int_0^\lambda i\,d\lambda = \int_0^\lambda \frac{\lambda}{L} d\lambda = \frac{1}{2}\frac{\lambda^2}{L}$$

$$= \frac{1}{2} \times \frac{(1.152)^2}{1.08} = 0.6144 \text{ J}$$

2.6 HYSTERESIS AND EDDY-CURRENT LOSSES

When a magnetic material undergoes cyclic magnetization, two kinds of power losses occur in it—hysteresis and eddy-current losses—which together are known as *core-loss*. The core-loss is important in determining heating, temperature rise, rating and efficiency of transformers, machines and other ac run magnetic devices.

Hysteresis Loss

Figure 2.18 shows a typical hysteresis loop of a ferromagnetic material. As the mmf is increased from zero to its maximum value, the energy stored in the field per unit volume of material is

$$\int_{-B_f}^{B_b = B_m} H dB = \text{area } ofabgo$$

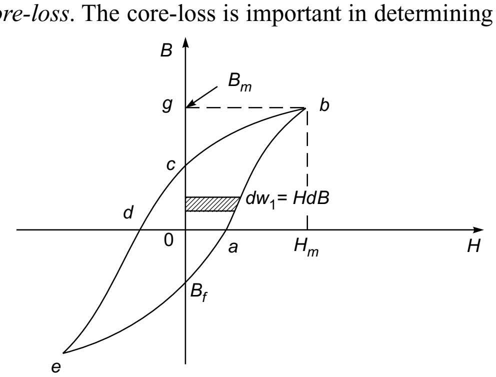

Fig. 2.18 *Hysteresis loss*

As H is now reduced to zero, dB being negative, the energy is given out by the magnetic field (from the exciting coil back to the voltage source) and has a value

$$\int_{B_b = B_m}^{B_c} H dB = \text{area } cbg$$

The net energy unrecovered in the process is area *ofabco* which is lost irretrievably in the form of heat and is called the hysteresis loss. The total hysteresis loss in one cycle is easily seen to be the area of the complete loop (*abcdefa*) and let it be indicated as w_h (hysteresis loss/unit volume). Then hysteresis loss in volume V of material when operated at f Hz is

$$P_h = w_h V f \quad \text{W} \tag{2.35}$$

In order to avoid the need for computation of the loop area, Steinmetz gave an empirical formula for computation of the hystersis loss based on experimental studies according to which

$$P_h = k_h f B_m^n \quad \text{W/m}^3 \tag{2.36}$$

where k_h is a characteristic constant of the core material, B_m is the maximum flux density and n, called the Steinmetz exponent, may vary from 1.5 to 2.5 depending upon the material and is often taken as 1.6.

Eddy-current Loss

When a magnetic core carries a time-varying flux, voltages are induced in all possible paths enclosing the flux. The result is the production of circulating currents in the core (all magnetic materials are conductors). These currents are known as eddy-currents and have power loss (i^2R) associated with them called eddy-current loss. This loss, of course, depends upon the resistivity of the material and lengths of the paths of circulating currents for a given cross-section. Higher resistivity and longer paths increase the effective resistance offered by the material to induced voltages resulting in reduction of eddy-current loss. High resistivity is achieved by adding silicon to steel and hence silicon steel is used for cores conducting alternating flux. Dividing up the material into thin *laminations* along the flow of flux, with each lamination lightly insulated (varnish is generally used) from the adjoining ones, increases the path length of the circulating currents with consequent reduction in eddy-current loss. The loss in fact can be shown to depend upon the square of lamination thickness. The lamination thickness usually varies from 0.3 to 5 mm for electromagnetic devices used in power systems and from about 0.01 to 0.5 mm for devices used in electronic applications where low core-loss is desired.

The eddy-current loss can be expressed by the empirical formula

$$p_e = k_e f^2 B^2 \quad \text{W}/m^3 \tag{2.37}$$

wherein

$$k_e = K_e' d^2/\rho \tag{2.38}$$

d being the thickness of lamination and ρ the resistivity of material.

It is only an academic exercise to split the core-loss into its two components. The core loss in fact arises from two types of flux variations: (i) flux that has a fixed axis and varies sinusoidally with time as in transformers (this is the type visualized in the above discussion), (ii) flux density is constant but the flux axis rotates. Actually in ac machines as well as in armature of dc machines the flux variation comprises both these types occurring simultaneously. The core-loss is measured experimentally on material specimen and presented graphically. Typical values of the *specific core-loss* (W/kg of material) are displayed in Figs 2.19 (a) and (b) for cold-rolled grain-oriented (crgos) steel. It is easy to see from these figures that for reasons mentioned above specific core loss is much higher in machines than in transformers.

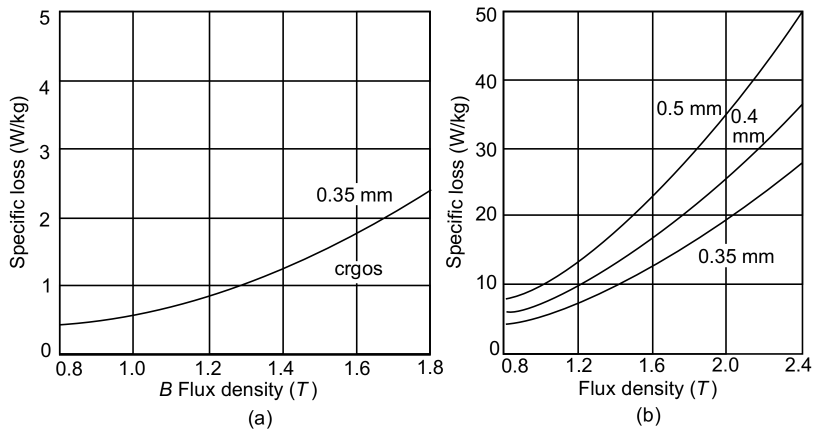

Fig. 2.19 *Core-loss at 50 Hz: (a) transformers, (b) machines*

EXAMPLE 2.9 *The total core loss of a specimen of silicon steel is found to be 1500 W at 50 Hz. Keeping the flux density constant the loss becomes 3000 W when the frequency is raised to 75 Hz. Calculate separately the hysteresis and eddy current loss at each of those frequencies.*

SOLUTION From Eqs. (2.36) and (2.37) for constant flux density, total core loss can be expressed as

$$P = Af + Bf^2 \quad \text{or} \quad P/f = A + Bf$$

$$1500/50 = A + 50\,B \quad \text{or} \quad 30 = A + 50\,B \qquad \text{(i)}$$

$$3000/75 = A + 75\,B \quad \text{or} \quad 40 = A + 75\,B \qquad \text{(ii)}$$

Solving Eqs. (i) and (ii), we get $A = 10$, $B = 2/5$

Therefore $$P = 10f + 2/5\, f^2 = P_h + P_e \qquad \text{(iii)}$$

At 50 Hz $$P_h = 10 \times 50 = 500 \text{ W}$$

$$P_e = 2/5 \times 2500 = 1000 \text{ W}$$

At 75 Hz $$P_h = 10 \times 75 = 750 \text{ W}$$

$$P_e = 2/5 \times (75)^2 = 2250 \text{ W}$$

2.7 PERMANENT MAGNETS

The permanent magnet is an important excitation source (life long) commonly employed for imparting energy to magnetic circuits used in rotating machines and other types of electromechanical devices. There are three classes of permanent magnet materials (or hard magnetic materials) used for permanent magnet dc (PMDC) motors: Alnicos, ceramics (ferrites) and rare-earth materials. Alnico magnets are used in motors up to 200 kW, while ceramic magnets are most economical in fractional kW motors. The rare-earth magnetic materials are very costly, but are the most economic choice in very small motors. Latest addition is neodymium-iron boron (Nd FeB). At room temperature, it has the highest energy product (to be explained later in this section) of all commonly available magnets. The high permeance and coercivity allow marked reductions in motor frame size for the same output compared to motors using ferrite (ceramic) magnets. For very high temperature applications Alnico or rare-earth cobalt magnets must be used.

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MAGNETIC CIRCUITS AND INDUCTION

2

2.1 INTRODUCTION

The electromagnetic system is an essential element of all rotating electric machinery and electromechanical devices as well as static devices like the transformer. The role of the electromagnetic system is to establish and control electromagnetic fields for carrying out conversion of energy, its processing and transfer. Practically all electric motors and generators, ranging in size from fractional horsepower units found in domestic appliances to the gigantic several thousand kW motors employed in heavy industry and several hundred megawatt generators installed in modern generating stations, depend upon the magnetic field as the coupling medium allowing interchange of energy in either direction between electrical and mechanical systems. A transformer though not an electromechanical conversion device, provides a means of transferring electrical energy between two electrical ports via the medium of a magnetic field. Further, transformer analysis runs parallel to rotating machine analysis and greatly aids in understanding the latter. It is, therefore, seen that all electric machines including transformers use the medium of magnetic field for energy conversion and transfer. The study of these devices essentially involves electric and magnetic circuit analysis and their interaction. Also, several other essential devices like relays, circuit breakers, etc. need the presence of a confined magnetic field for their operation.

The purpose of this chapter is to review the physical laws governing magnetic fields, induction of emf and production of mechanical force, and to develop methods of magnetic-circuit analysis. Simple magnetic circuits and magnetic materials will be briefly discussed. In the chapters to follow, how the concepts of this chapter are applied in the analysis of transformers and machines will be shown.

2.2 MAGNETIC CIRCUITS

The exact description of the magnetic field is given by the Maxwell's equations* and the constitutive relationship of the medium in which the field is established.

---

* Maxwell's equations governing the electric and magnetic fields are

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{and} \quad \vec{D} = \varepsilon_0 \vec{E}\,;\ \varepsilon_0 = 8.85 \times 10^{-12}$$

wherein $\vec{J}$ = conduction current density and $\vec{D}$ = displacement current density, negligible for slowly-varying fields $\vec{D} = \varepsilon_0 \vec{E}$; $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m).

Such description apart from being highly complex is otherwise not necessary for use in electric machines wherein the fields (magnetic and electric) are slowly varying (fundamental frequency being 50 Hz) so that the *displacement current* can be neglected. The magnetic field can then be described by Ampere's law and is solely governed by the *conduction current*. This law is in integral form and is easily derivable from the third Maxwell's equation (by ignoring displacement current) by means of well-known results in vector algebra. The Ampere's law is reproduced as follows:

$$\int_s \vec{J} \cdot d\vec{s} = \oint \vec{H} \cdot d\vec{l} \tag{2.1}$$

wherein $\vec{J}$ = conduction current density

$\vec{H}$ = magnetic field intensity

s = the surface enclosed by the closed path of length l

$d\vec{s}$ = differential surface

$d\vec{l}$ = differential length

Consider the example of a simple electromagnetic system comprising an exciting coil and *ferromagnetic* core as shown in Fig. 2.1. The coil has N turns and carries a constant (dc) current of i A. The magnetic field is established in the space wherein most of the total magnetic flux set up is confined to the ferromagnetic core for reasons which will soon become obvious. Consider the flux path through the core (shown dotted) which in fact is the *mean path* of the core flux. The total current piercing the surface enclosed by this path is as follows:

$$\int_s \vec{J} \cdot d\vec{s} = Ni$$

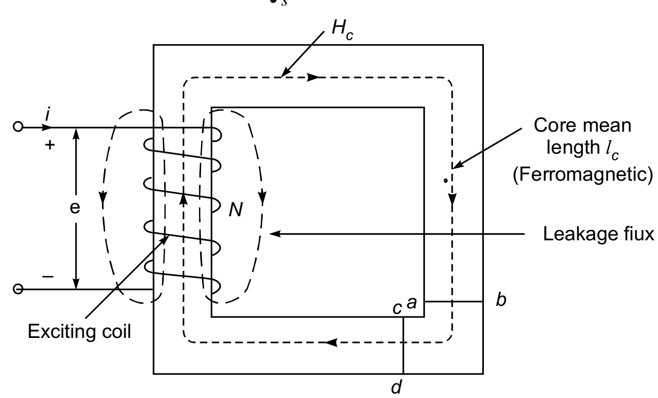


Fig. 2.1 *A simple magnetic system*

Hence Eq. (2.1) acquires the form

$$Ni = \oint_l \vec{H} \cdot d\vec{l} \tag{2.2}$$

Since N is the number of coil turns and i the exciting current in amperes, the product $\mathcal{F} = Ni$ has the units of *ampere-turns* (AT) and is the cause of establishment of the magnetic field. It is known as the *magnetomotive force* (mmf) in analogy to the electromotive force (emf) which establishes current in an electric circuit.

The magnetic field intensity H causes a flux density B to be set up at every point along the flux path which is given by

$$B = \mu H = \mu_0 \mu_r H \qquad \text{(for flux path in core)} \tag{2.3a}$$

and

$$B = \mu_0 H \qquad \text{(for flux path in air)} \tag{2.3b}$$

The units of flux density are weber (Wb)/m^2 called *tesla* (T). The term μ_0 is the *absolute permeability* of free space and has a value of

$$\mu_0 = 4\pi \times 10^{-7} \qquad \text{henry (H)/m}$$

The permeability $\mu = \mu_0 \mu_r$ of a material medium is different from μ_0 because of a certain phenomenon occurring in the material. The term μ_r is referred to as *relative permeability* of a material and is in the range of 2000-6000 for ferromagnetic materials (see Sec. 2.3). It is, therefore, seen that for a given H, the flux density B and, therefore, the flux over a given area

$$\phi = \oint_s \vec{B} \cdot d\vec{s}$$

will be far larger in the magnetic core in Fig. 2.1 than in the air paths. Hence, it is safe to assume that the magnetic flux set up by mmf Ni is mainly confined to the ferromagnetic core and the flux set up in air paths is of negligible value. The flux set up in air paths is known as the *leakage flux* as if it leaks through the core; some of the leakage flux paths are shown chain-dotted in Fig. 2.1. There is no way to avoid magnetic leakage as there are no magnetic insulators in contrast to electric insulators which confine the electric current to the conductor for all practical purposes. The effect of the leakage flux is incorporated in machine models through the concept of the *leakage inductance*.

The direction of field intensity *is* H and so the direction of flux ϕ is determined from the Right Hand Rule (RHR). It is stated as:

Imagine that you are holding a current carrying conductor in your right hand with the thumb pointing in the direction of current. Then the direction in which the fingers curl gives the direction of flux. In case of a coil you imagine that you are grasping the coil in right hand with the thumb in the direction of current; then the fingers curl in the direction of flux.

The reader may apply RHR to the exciting coil in Fig. 2.1 to verify the direction of flux as shown in the figure.

The magnetic field intensity $\vec{H}$ is tangential to a flux line all along its path, so that the closed vector integration in Eq. (2.2) along a flux-line reduces to closed scalar integration, i.e.

$$Ni = \oint_l H \cdot dl \tag{2.4}$$

With the assumption of negligible leakage flux, the flux piercing the core cross-section at any point remains constant. Further, from the consideration of symmetry it immediately follows that the flux density over straight parts of the core is uniform at each cross-section and remains constant along the length; such that H is constant along the straight parts of the core. Around the corners, flux lines have different path lengths between magnetic equipotential planes (typical ones being *ab* and *cd* shown in Fig. 2.1) so that H varies from a high value along inner paths to a low value along outer paths. It is reasonable to assume that H shown dotted along the mean path will have the same value as in straight parts of the core (this *mean path technique* renders simple the analysis of magnetic circuits of machines and transformers).

It has been seen previously that the magnetic field intensity along the mean flux path in the core can be regarded constant at H_c. It then follows from Eq. (2.4) that

$$\mathcal{F} = Ni = H_c l_c \tag{2.5}$$

where $\mathcal{F}$ = **mmf** in AT and l_c = mean core length (m)

From Eq. (2.5)

$$H_c = \frac{Ni}{l_c} \quad \text{AT/m} \tag{2.6}$$

If one now imagines that the exciting current i varies with time, Eq. (2.6) would indicate that H_c will vary in unison with it. Such fields are known as *quasi-static* fields in which the field pattern in space is fixed but the field intensity at every point varies as a replica of the time variation of current. This simplified field picture is a consequence of negligible displacement current in slowly-varying fields as mentioned earlier. In a quasi-static field, the field pattern and field strength at a particular value of time-varying exciting current will be the same as with a direct current of that value. In other words, a field problem can be solved with dc excitation and then any time variation can be imparted to it.

Now, the core *flux density* is given by,

$$B_c = \mu_c H_c \quad \text{tesla (T)}$$

and core flux (assumed to be total flux) is given by,

$$\phi = \oint_s \vec{B} \cdot d\vec{s} = B_c A_c \quad \text{Wb}$$

where A_c = cross-sectional area of core and flux in the limbs is oriented normal to cross-sectional area. Then from Eq. (2.6)

$$\phi = \mu_c H_c A_c = \frac{Ni}{\left(\frac{l_c}{\mu_c A_c}\right)} \quad \text{or} \quad \phi = \frac{\mathcal{F}}{\mathcal{R}} = \mathcal{F}\mathcal{P} \tag{2.7}$$

where
$$\mathcal{R} = \frac{\mathcal{F}}{\phi} = \frac{l_c}{\mu_c A_c} = \textbf{\textit{reluctance}}^* \text{ of the magnetic circuit (AT/Wb)} \tag{2.8}$$

and $\mathcal{P} = 1/\mathcal{R}$ = *permeance* of the magnetic circuit. It is, therefore, seen that by certain simplifying assumptions and field symmetries, it has been possible to lump the distributed magnetic system into a lumped magnetic circuit described by Eq. (2.7) which is analogous to Ohm's law in dc circuits. The electrical circuit analog of the magnetic system (now reduced to a magnetic circuit) is shown in Fig. 2.2 wherein $\mathcal{F}$ (mmf) is analogous to E (emf), $\mathcal{R}$ (reluctance) is analogous to R (resistance) and ϕ (flux) is analogous to i (current).

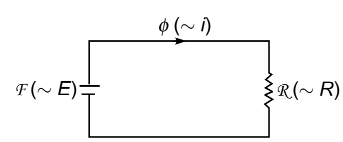


Fig. 2.2 *Electrical analog of the simple magnetic circuit of Fig. 2.1*

The analogy though useful is, however, not complete; there being two points of difference: (i) magnetic reluctance is nondissipative of energy unlike electric resistance, (ii) when $\mathcal{F}$ is time-varying, the magnetic circuit still remains resistive as in Fig. 2.2, while inductive effects are bound to appear in an electric circuit. This is because there is no time-lag between the exciting current and the establishment of magnetic flux (quasi-static field).

* Unit of reluctance is AT/Wb and will not be specified every time in examples.

The lumped magnetic circuit and its electrical analog are useful concepts provided the permeability (μ) of the core material and, therefore, the core reluctance is constant as is tacitly assumed above. This, however, is not the case with ferromagnetic materials, but when air-gaps are involved, the assumption of constant reluctance is generally valid and leads to considerable simplicity in magnetic circuit analysis.

In more complicated magnetic circuits—with multiple excitations and series-parallel core arrangement—the general theorems of electric circuits apply, i.e. Kirchhoff's voltage (mmf) law and Kirchhoff's current (flux) law. This is illustrated in Example 2.3.

B-H Relationship (Magnetization Characteristic)

In free space (also nonmagnetic materials), the permeability μ_0 is constant so that *B-H* relationship is linear. This, however, is not the case with ferromagnetic materials used in electric machines, wherein the *B-H* relationship is strictly nonlinear in two respects—*saturation* and *hysteresis*. Hysteresis non-linearity is the double valued *B-H* relationship exhibited in cyclic variation of *H* (i.e. exciting current). This nonlinearity is usually ignored in magnetic circuit calculations and is important only when current wave shape and power loss are to be accounted for. This is discussed in Sections 2.3 and 2.6. A typical normal *B-H* relationship (magnetization characteristic) for ferromagnetic materials is shown in Fig. 2.3. It has an initial nonlinear zone, a middle almost linear zone and a final saturation zone in which *B* progressively increases less rapidly with *H* compared to the linear zone. In the deep saturation zone, the material behaves like free space.

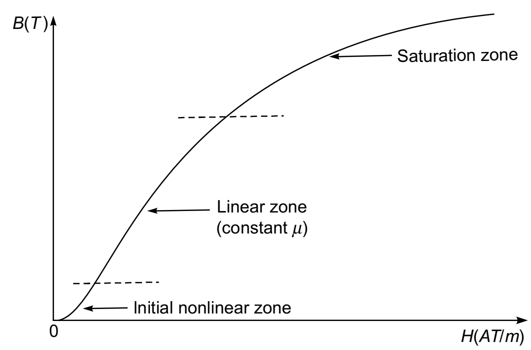


Fig. 2.3 *Typical normal magnetization curve of ferromagnetic material*

Due to considerations dictated by economy, electric machines and transformers are designed such that the magnetic material is slightly saturated (i.e. somewhat above the linear zone). In exact magnetic circuit calculations the nonlinear magnetization curve has to be used necessitating graphical/numerical solutions.

Core with Air-gap

Transformers are wound on closed cores as in Fig. 2.1. Rotating machines have a moving element and must therefore have air-gaps in the cores out of necessity. A typical magnetic circuit with an air-gap is shown in Fig. 2.4. It is assumed that the air-gap is narrow and the flux coming out of the core passes straight down the air-gap such that the flux density in the air-gap is the same as in the core. Actually as will soon be seen, that the flux in the gap fringes out so that the gap flux density is somewhat less than that of the core. Further, let the core permeability μ_c be regarded as constant (linear magnetization characteristic).

The mmf *Ni* is now consumed in the core plus the air-gap. From the circuit model of Fig. 2.4(b) or directly from Fig. 2.4(a)

$$Ni = H_c l_c + H_g l_g \tag{2.9a}$$

or

$$Ni = \frac{B_c}{\mu_c} l_c + \frac{B_g}{\mu_0} l_g \tag{2.9b}$$

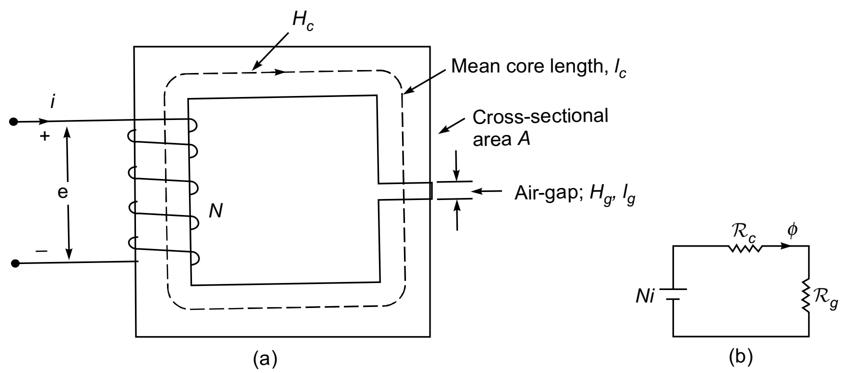


Fig. 2.4 *A typical magnetic circuit with air-gap and its equivalent electric circuit*

Assuming that all the core flux passes straight down the air-gap (it means no fringing (see Fig. 2.5))

$$B_g = B_c$$

$\therefore$
$$\phi = B_c A = B_g A \tag{2.10}$$

Substituting Eq. (2.10) in Eq. (2.9b)

$$Ni = \phi\left(\frac{l_c}{\mu_c A}\right) + \phi\left(\frac{l_g}{\mu_0 A}\right) \tag{2.11}$$

Recognizing various quantities in Eq. (2.11)

$$\mathcal{F} = \phi(\mathcal{R}_c + \mathcal{R}_g) = \phi\mathcal{R}_{eq} \tag{2.12}$$

where
$$\mathcal{R}_c = \frac{l_c}{\mu_c A} = \text{core reluctance}$$

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A} = \text{air-gap reluctance}$$

From Eq. (2.12)

$$\phi = \frac{\mathcal{F}}{\mathcal{R}_c + \mathcal{R}_g} = \frac{\mathcal{F}/\mathcal{R}_g}{1 + \mathcal{R}_c/\mathcal{R}_g} \tag{2.13}$$

But
$$\frac{\mathcal{R}_c}{\mathcal{R}_g} = \frac{\mu_0 l_c}{\mu_c l_g} \ll 1$$

because μ_c is 2000 to 6000 times μ_0 in ferromagnetic materials. The permeability effect predominates the usual core and air-gap dimensions even though $l_c \gg l_g$. It then follows from Eq. (2.13), that

$$\phi \approx \mathcal{F}/\mathcal{R}_g \tag{2.14}$$

which means that in a magnetic circuit with air-gap(s), core reluctance may be neglected with no significant loss of accuracy. This assumption will be generally made in modelling rotating machines. The effect of core saturation (reduction of core permeability) will be introduced as a correction wherever greater accuracy is desired.

Magnetic Circuit Calculations

Normally magnetic circuit calculations involve two types of problems. In the first type of problem it is required to determine the excitation (mmf) needed to establish a desired flux or flux density at a given point in a magnetic circuit. This is the normal case in designing electromechanical devices and is a straight forward problem. In the second category, the flux (or flux density) is unknown and is required to be determined for a given geometry of the magnetic circuit and specified mmf. This kind of problem arises in magnetic amplifiers wherein this resultant flux is required to be determined owing to the given excitation on one or more control windings. A little thought will reveal that there is no direct analytical solution to this problem because of the non-linear *B-H* characteristic of the magnetic material. Graphical/numerical techniques have to be used in obtaining the solution of this problem.

Leakage Flux

In all practical magnetic circuits, most of the flux is confined to the intended path by use of magnetic cores but a small amount of flux always leaks through the surrounding air. This stray flux as already stated is called the *leakage flux*, Leakage is characteristic of all magnetic circuits and can never be fully eliminated. Calculations concerning the main magnetic circuit are usually carried out with the effect of leakage flux either ignored or empirically accounted for. Special studies of leakage must be made for ac machines and transformers since their performance is affected by it.

Fringing

At an air-gap in a magnetic core, the flux fringes out into neighbouring air paths as shown in Fig. 2.5; these being of reluctance comparable to that of the gap. The result is *nonuniform* flux density in the air-gap (decreasing outward), enlargement of the effective air-gap area and a decrease in the average gap flux density. The fringing effect also disturbs the core flux pattern to some depth near the gap. The effect of fringing increases with the air-gap length. Corrections for fringing in short gaps (as used in machines) are empirically made by adding one gap length to each of the two dimensions making up its area. For the example of the core with the air-gap previously presented, the gap reluctance would now be given by

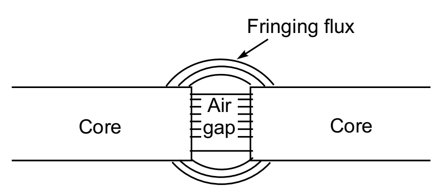


Fig. 2.5 *Flux fringing at air-gap*

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_g}$$

which will be less than the previous value as $A_g > A$.

It can be shown theoretically that the magnetic flux leaves and enters the surface of an infinitely permeable material normally. This will be nearly so in ferromagnetic materials which have high permeability. In electric machines a small amount of the tangential flux component present at iron surfaces will be neglected.

Stacking Factor

Magnetic cores are made up of thin, lightly insulated (coating of varnish) *laminations* to reduce power loss in cores due to the eddy-current phenomenon (explained in Sec. 2.6). As a result, the net cross-sectional area of the core occupied by the magnetic material is less than its gross cross-section; their ratio (less than unity) being known as the stacking factor. Depending upon the thickness of laminations, stacking factor may vary from 0.5–0.95, approaching unity as the lamination thickness increases.

EXAMPLE 2.1 *The magnetic circuit of Fig. 2.4(a) has dimensions:* $A_c = 4 \times 4$ *cm*^{2}, $l_g = 0.06$ *cm,* $l_c = 40$ *cm;* $N = 600$ *turns. Assume the value of* $\mu_r = 6000$ *for iron. Find the exciting current for* $B_c = 1.2$ *T and the corresponding flux and flux linkages.*

SOLUTION From Eq. (2.9), the ampere-turns for the circuit are given by

$$Ni = \frac{B_c}{\mu_0 \mu_r} l_c + \frac{B_g}{\mu_0} l_g \qquad \text{(i)}$$

Neglecting fringing

$$A_c = A_g \quad \text{therefore} \quad B_c = B_g$$

Then

$$i = \frac{B_c}{\mu_0 N}\left(\frac{l_c}{\mu_r} + l_g\right)$$

$$= \frac{1.2}{4\pi \times 10^{-7} \times 600}\left(\frac{40}{6000} + 0.06\right) \times 10^{-2} \qquad \text{(ii)}$$

$$= 1.06 \text{ A}$$

The reader should note that the reluctance of the iron path of 40 cm is only $\left(\frac{2/3}{6}\right) = 0.11$ of the reluctance of the 0.06 cm air-gap.

$$\phi = B_c A_c = 1.2 \times 16 \times 10^{-4} = 19.2 \times 10^{-4} \text{ Wb}$$

Flux linkages,

$$\lambda = N\phi = 600 \times 19.2 \times 10^{-4} = 1.152 \text{ Wb-turns}$$

If fringing is to be taken into account, one gap length is added to each dimension of the air-gap constituting the area. Then

$$A_g = (4 + 0.06)\,(4 + 0.06) = 16.484 \text{ cm}^2$$

Effective $A_g > A_c$ reduces the air-gap reluctance. Now

$$B_g = \frac{19.2 \times 10^{-4}}{16.484 \times 10^{-4}} = 1.165 \text{ T}$$

From Eq. (i)

$$i = \frac{1}{\mu_0 N}\left(\frac{B_c l_c}{\mu_r} + B_g l_g\right) \qquad \text{(iii)}$$

$$= \frac{1}{4\pi \times 10^{-7} \times 600}\left(\frac{1.2 \times 40 \times 10^{-2}}{6000} + 1.165 \times 0.06 \times 10^{-2}\right)$$

$$= 1.0332 \text{ A}$$

EXAMPLE 2.2 *A wrought iron bar 30 cm long and 2 cm in diameter is bent into a circular shape as shown in Fig. 2.6. It is then wound with 600 turns of wire. Calculate the current required to produce a flux of 0.5 mWb in the magnetic circuit in the following cases:*

(i) no air-gap;

(ii) with an air-gap of 1 mm; μ_r *(iron) = 4000 (assumed constant); and*

(iii) with an air-gap of 1 mm; assume the following data for the magnetization of iron:

H in AT/m	*2500*	*3000*	*3500*	*4000*
B in T	*1.55*	*1.59*	*1.6*	*1.615*

SOLUTION

(i) No air-gap

$$\mathcal{R}_c = \frac{30 \times 10^{-2}}{4000 \times 4\pi \times 10^{-7} \times \pi \times 10^{-4}} = 1.9 \times 10^5$$

$$Ni = \phi \mathcal{R}_c$$

or $$i = \phi \mathcal{R}_c / N = \frac{0.5 \times 10^{-3} \times 1.9 \times 10^5}{600} = 0.158 \text{ A}$$

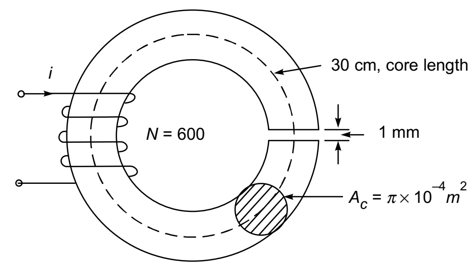


Fig. 2.6

(ii) Air-gap = 1 mm, μ_r (iron) = 4000

$$\mathcal{R}_c = 1.9 \times 10^5 \text{ (as in part (i)}$$

$$\mathcal{R}_g = \frac{1 \times 10^{-3}}{4\pi \times 10^{-7} \times \pi \times 10^{-4}} = 25.33 \times 10^5$$

$$\mathcal{R}(\text{total}) = \mathcal{R}_c + \mathcal{R}_g = 27.1 \times 10^5$$

$$\therefore \quad i = \frac{0.5 \times 10^{-3} \times 27.1 \times 10^5}{600} = 2.258 \text{A}$$

(iii) Air-gap = 1 mm; *B-H* data as given

$$B_c = B_g = \frac{0.5 \times 10^{-3}}{\pi \times 10^{-4}} = 1.59 \text{ T (fringing neglected)}$$

$$H_g = \frac{B_g}{\mu_0} = \frac{1.59}{4\pi \times 10^{-7}}$$

$$\text{AT}_g = H_g l_g = \frac{1.59 \times 1 \times 10^{-3}}{4\pi \times 10^{-7}} = 1265$$

From the given magnetization data (at $B_c = 1.59$ T),

$$H_c = 3000 \text{ AT/m}$$

$$\text{AT}_c = H_c l_c = 3000 \times 30 \times 10^{-2} = 900$$

$$\text{AT (total)} = \text{AT}_c + \text{AT}_g = 900 + 1265 = 2165$$

$$i = \frac{2165}{600} = 3.61 \text{ A}$$

EXAMPLE 2.3 *The magnetic circuit of Fig. 2.7 has cast steel core with dimensions as shown:*

Mean length from A to B through either outer limb = 0.5 m

Mean length from A to B through the central limb = 0.2 m

In the magnetic circuit shown it is required to establish a flux of 0.75 mWb in the air-gap of the central limb. Determine the mmf of the exciting coil if for the core material (a) $\mu_r = \infty$ (b) $\mu_r = 5000$. Neglect fringing.

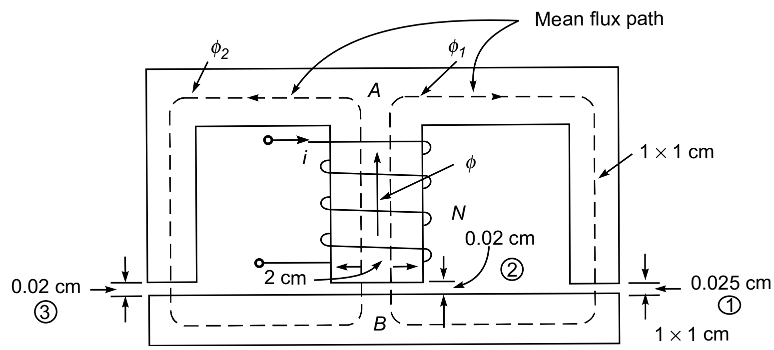


Fig. 2.7

SOLUTION

(a) $\mu_r = \infty$, i.e. there are no mmf drops in the magnetic core. It is easy to see from Fig. 2.7 that the two outer limbs present a parallel magnetic circuit. The electrical analog of the magnetic circuit is drawn in Fig. 2.8(a). Various gap reluctances are:

$$\mathcal{R}_{g1} = \frac{0.025\times 10^{-2}}{4\pi\times 10^{-7}\times 1\times 10^{-4}} = 1.99\times 10^6$$

$$\mathcal{R}_{g2} = \frac{0.02\times 10^{-2}}{4\pi\times 10^{-7}\times 1\times 10^{-4}}\ 1.592\times 10^6$$

$$\mathcal{R}_{g3} = \frac{0.02\times 10^{-2}}{4\pi\times 10^{-7}\times 2\times 10^{-4}} = 0.796\times 10^6$$

From Fig. 2.8(b),

$$\begin{aligned} Ni &= 0.75\times 10^{-3}\,(\mathcal{R}_{g3} + \mathcal{R}_{g1} \parallel \mathcal{R}_{g2}) \\ &= 0.75\times 10^{-3}\,(0.796 + 0.844)\times 10^6 \\ &= 1230 \text{ AT} \end{aligned}$$

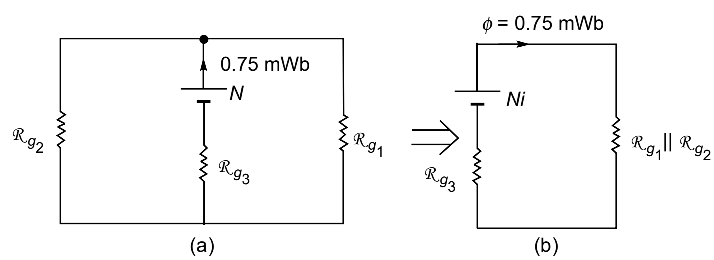


Fig. 2.8 *Electrical analog of Fig. 2.7*

(b) $\mu_r = 5000$. This means that the reluctance of magnetic core must be taken into consideration. The analogous electric circuit now becomes that of Fig. 2.9.

Since gap lengths are negligible compared to core lengths, various core reluctances can be calculated as follows:

$$\mathcal{R}_{c1} = \frac{0.5}{4\pi \times 10^{-7} \times 5000 \times 1 \times 10^{-4}} = 0.796 \times 10^6$$

$$\mathcal{R}_{c2} = \mathcal{R}_{c1} = 0.796 \times 10^6$$

$$\mathcal{R}_{c3} = \frac{0.2}{4\pi \times 10^{-7} \times 5000 \times 2 \times 10^{-4}} = 0.159 \times 10^6$$

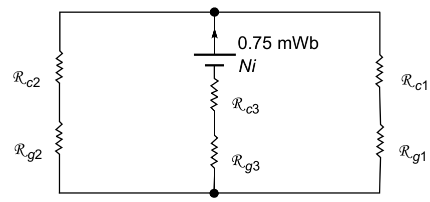


Fig. 2.9

The equivalent reluctance is

$$\mathcal{R}_{eq} = (\mathcal{R}_{c1} + \mathcal{R}_{g1}) \| (\mathcal{R}_{c2} + \mathcal{R}_{g2}) + \mathcal{R}_{c3} + \mathcal{R}_{g3})$$

$$= \frac{27.86 \times 23.86}{51.72} \times 10^6 + 0.955 \times 10^6 = 1.955 \times 10^6$$

Now $Ni = \phi \mathcal{R}_{eq}$

$$= 0.75 \times 10^{-3} \times 1.955 \times 10^6$$

$$= 1466 \text{ AT}$$

EXAMPLE 2.4 *The magnetic circuit of Fig. 2.10 has cast steel core. The cross-sectional area of the central limb is 800 mm² and that of each outer limb is 600 mm². Calculate the exciting current needed to set up a flux of 0.8 mWb in the air gap. Neglect magnetic leakage and fringing. The magnetization characteristic of cast steel is given in Fig. 2.16.*

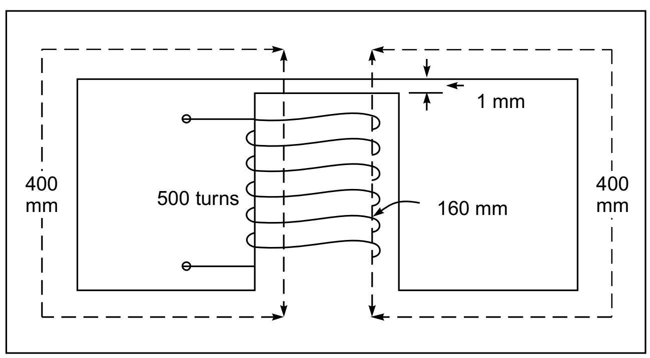


Fig. 2.10

SOLUTION

Air gap $B_g = \frac{0.8}{800} \times \frac{10^{-3}}{10^{-6}} = 1 \text{ T}$ and $H_g = \frac{1}{4\pi \times 10^{-7}}$ AT/m

$$\mathcal{F}_g = \frac{1}{4\pi \times 10^{-7}} \times 1 \times 10^{-3} = 796 \text{ AT}$$

Central limb $B_c = B_g = 1 \text{ T}$

From Fig. 2.16 $H_c = 1000$ AT/m

$$\mathcal{F}_c = 1000 \times 160 \times 10^{-3} = 160 \text{ AT}$$

Because of symmetry, flux divides equally between the two outer limbs. So

$$\phi \text{ (outer limb)} = 0.8/2 = 0.4 \text{ mWb}$$

$$B \text{ (outer limb)} = \frac{0.4 \times 10^{-3}}{600 \times 10^{-6}} = 0.667 \text{ AT}$$

$$\mathcal{F} \text{ (outer limb)} = 375 \times 400 \times 10^{-3} = 150 \text{ AT}$$

$$\mathcal{F} \text{ (total)} = 796 + 160 + 150 = 1106 \text{ AT}$$

$$\text{Exciting current} = 1106/500 = 2.21 \text{ A}$$

EXAMPLE 2.5 *The magnetic circuit of Fig. 2.11 has a cast steel core whose dimensions are given below:*

Length (ab + cd) = 50 cm	*Cross-sectional area = 25 cm²*
Length ad = 20 cm	*Cross-sectional area = 12.5 cm²*
Length dea = 50 cm	*Cross-sectional area = 25 cm²*

Determine the exciting coil mmf required to establish an air-gap flux of 0.75 m Wb. Use the B-H curve of Fig. 2.16.

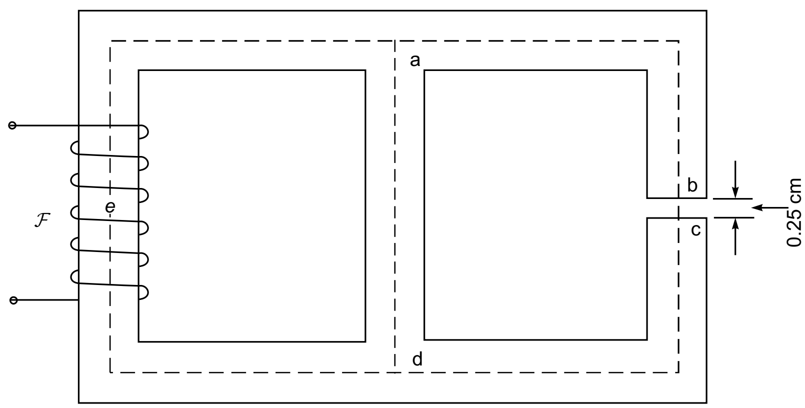


Fig. 2.11

SOLUTION Assuming no fringing the flux density in the path *abcd* will be same, i.e.

$$B = \frac{0.75 \times 10^{-3}}{25 \times 10^{-4}} = 0.3 \text{ T}$$

$$\mathcal{F}_{bc} = \frac{B}{\mu_0} l_{bc} = \frac{0.3 \times 0.25 \times 10^{-3}}{4\pi \times 10^{-7}} = 60 \text{ AT}$$

$H_{ab} = H_{cd}$ (from Fig. 2.16 for cast steel for $B = 0.3$ T) = 200 AT/m

$$\mathcal{F}_{ab+cd} = 200 \times 50 \times 10^{-2} = 100 \text{ AT}$$

$$\therefore \quad \mathcal{F}_{ad} = 60 + 100 = 160 \text{ AT}$$

$$H_{ad} = \frac{160}{20 \times 10^{-2}} = 800 \text{ AT/m}$$

$$B_{ad} \text{ (from Fig. 2.16)} = 1.04 \text{ T}$$

$$\phi_{ad} = 1.04 \times 12.5 \times 10^{-4} = 1.3 \text{ mWb}$$

$$\phi_{dea} = 0.75 + 1.3 = 2.05 \text{ mWb}$$

$$B_{dea} = \frac{2.05 \times 10^{-3}}{25 \times 10^{-4}} = 0.82 \text{ T}$$

$$H_{dea} \text{ (from Fig. 2.16)} = 500 \text{ A T/m}$$

$$\mathcal{F}_{dea} = 500 \times 50 \times 10^{-2} = 250 \text{ AT}$$

$$\mathcal{F} = \mathcal{F}_{dea} + \mathcal{F}_{ad} = 250 + 160 = 410 \text{ AT}$$

EXAMPLE 2.6 *A cast steel ring has a circular cross-section of 3 cm in diameter and a mean circumference of 80 cm. A 1 mm air-gap is cut out in the ring which is wound with a coil of 600 turns.*

(a) Estimate the current required to establish a flux of 0.75 mWb in the air-gap. Neglect fringing and leakage.

(b) What is the flux produced in the air-gap if the exciting current is 2 A? Neglect fringing and leakage.

Magnetization data:

H (AT/m)	*200*	*400*	*600*	*800*	*1000*	*1200*	*1400*	*1600*	*1800*	*2020*
B (T)	*0.10*	*0.32*	*0.60*	*0.90*	*1.08*	*1.18*	*1.27*	*1.32*	*1.36*	*1.40*

SOLUTION

$$\phi = 0.75 \times 10^{-3} \text{ Wb}$$

$$B_g = \phi/A = \frac{0.75 \times 10^{-3}}{\pi \times \left(\frac{0.03}{2}\right)^2} = 1.06 \text{ T}$$

$$B_c = B_g \text{ (no fringing)}$$

Reading from the *B-H* curve drawn in Fig. 2.12,

$$H_c = 900 \text{ AT/m}$$

$$l_c = 0.8 \text{ m (air-gap length can be neglected)}$$

$$AT_c = H_c l_c = 900 \times 0.8 = 720$$

$$AT_g = \frac{1.06}{4\pi \times 10^{-7}} \times 10^{-3} = 843$$

$$Ni = AT_c + AT_g = 720 + 843 = 1563$$

Therefore $i = \dfrac{1563}{600} = 2.6\text{A}$

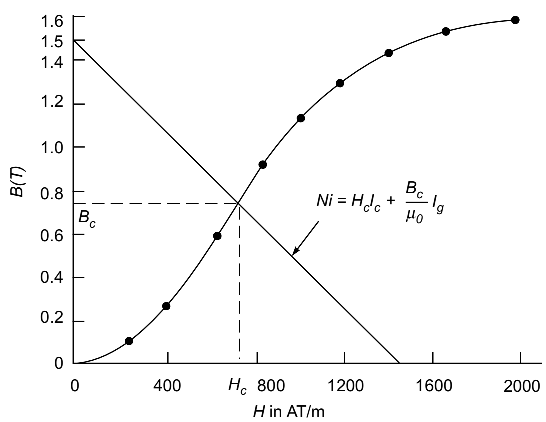


Fig. 2.12

(b) The excitation is now given and the flux is to be determined from the *B-H* curve given. The problem must, therefore, be solved numerically/graphically. It is solved here graphically. Now

$$Ni = \frac{B_2}{\mu_0} l_g + H_c l_c; \ (B_g = B_c) \tag{i}$$

This is a linear equation in B_c and H_c; the second equation is the nonlinear *B-H* curve. The intersection of the two for a given *Ni* will yield the solution. For this problem

$$Ni = 600 \times 2 = 1200 \text{ AT}$$

Substituting various values in Eq. (i)

$$1200 = \frac{B_c}{4\pi \times 10^{-7}} \times 10^{-3} + 0.8\, H_c \qquad \text{(ii)}$$

This equation is plotted in Fig. 2.12, by locating the points

$$H_c = 0, \quad B_c = 1.5$$
$$B_c = 0, \quad H_c = 1500$$

The intersection gives the result

$$B_c = 0.78 \text{ T}$$
$$\phi = B_c A = 0.78 \times \frac{\pi}{4} (0.03)^2 = 0.55 \text{ mWb}$$

2.3 MAGNETIC MATERIALS AND THEIR PROPERTIES

From the magnetic point of view* a material is classified according to the nature of its relative permeability (μ_r). All nonmagnetic materials are classified as *paramagnetic*, μ_r slightly greater than 1, and *diamagnetic*, μ_r slightly less than 1. For all practical purposes, μ_r of these materials can be regarded as unity, i.e. their magnetic properties are very much similar to that of free space. Such materials are not of interest to us in this treatise.

Materials which are of interest to us are those whose relative permeability is much higher than that of free space. These can be classified as *ferromagnetic* and *ferrimagnetic*. Ferromagnetic materials can be further subdivided as *hard* and *soft*. Hard ferromagnetic materials include permanent magnet materials, such as alnicos, chromium steels, certain copper-nickel alloys and several other metal alloys. Soft ferromagnetic materials are iron and its alloys with nickel, cobalt, tungsten and aluminium. Silicon steels and cast steels are the most important ferromagnetic materials for use in transformers and electric machines. Ferrimagnetic materials are the *ferrites* and are composed of iron oxides—MeO. Fe_2O_3, where Me represents a metallic ion. Ferrites are also subgrouped as hard (permanent magnetic) and soft (nickel-zinc and manganese-zinc) ferrites. Soft ferrites are quite useful in high frequency transformers, microwave devices, and other similar high-frequency operations. There is a third category of magnetic materials, known as *superparamagnetic*, made from powdered iron or other magnetic particles. These materials are used in transformers for electronics and cores for inductors. *Permalloy* (molybdenum-nickel-iron powder) is the best known example of this important category of magnetic materials.

Properties of Magnetic Materials

Magnetic materials are characterized by high permeability and the nonlinear *B-H* relationship which exhibits both saturation and hysteresis. The physics of these properties is explained by the domain theory of magnetization usually taught in junior level courses.

The *B-H* relationship for cyclic *H* is the *hysteresis loop* shown in Fig. 2.13 for two values of maximum flux density. It is easily observed from this figure that *B* is a symmetrical two-valued function of *H*; at any given *H*, *B* is higher if *H* is reducing compared to when *H* is increasing. This is the basic hysteresis property in which *B* lags behind *H*. It can also be recognized as a *memory-type non-linearity* in which the material

* For the theory of magnetization based on atomic structure of materials a suitable book on material science may be consulted.

7 Magnetic circuits

At the end of this chapter you should be able to:

- describe the magnetic field around a permanent magnet
- state the laws of magnetic attraction and repulsion for two magnets in close proximity
- define magnetic flux, Φ, and magnetic flux density, B, and state their units
- perform simple calculations involving $B = \frac{\Phi}{A}$
- define magnetomotive force, F_m, and magnetic field strength, H, and state their units
- perform simple calculations involving $F_m = NI$ and $H = \frac{NI}{l}$
- define permeability, distinguishing between μ_0, μ_r and μ
- understand the B–H curves for different magnetic materials
- appreciate typical values of μ_r
- perform calculations involving $B = \mu_0 \mu_r H$
- define reluctance, S, and state its units
- perform calculations involving $S = \frac{\text{mmf}}{\Phi} = \frac{l}{\mu_0 \mu_r A}$
- perform calculations on composite series magnetic circuits
- compare electrical and magnetic quantities
- appreciate how a hysteresis loop is obtained and that hysteresis loss is proportional to its area

7.1 Magnetic fields

A **permanent magnet** is a piece of ferromagnetic material (such as iron, nickel or cobalt) which has properties of attracting other pieces of these materials. A permanent magnet will position itself in a north and south direction when freely suspended. The north-seeking end of the magnet is called the **north pole**, ***N***, and the south-seeking end the **south pole**, ***S***.

The area around a magnet is called the **magnetic field** and it is in this area that the effects of the **magnetic force** produced by the magnet can be detected. A magnetic field cannot be seen, felt, smelt or heard and therefore is difficult to represent. Michael Faraday suggested that the magnetic field could be represented pictorially, by imagining the field to consist of **lines of magnetic flux**, which enables investigation of the distribution and density of the field to be carried out.

The distribution of a magnetic field can be investigated by using some iron filings. A bar magnet is placed on a flat surface covered by, say,

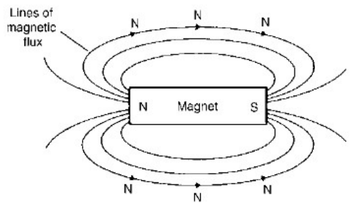


Figure 7.1

cardboard, upon which is sprinkled some iron filings. If the cardboard is gently tapped the filings will assume a pattern similar to that shown in Figure 7.1. If a number of magnets of different strength are used, it is found that the stronger the field the closer are the lines of magnetic flux and vice versa. Thus a magnetic field has the property of exerting a force, demonstrated in this case by causing the iron filings to move into the pattern shown. The strength of the magnetic field decreases as we move away from the magnet. It should be realized, of course, that the magnetic field is three dimensional in its effect, and not acting in one plane as appears to be the case in this experiment.

If a compass is placed in the magnetic field in various positions, the direction of the lines of flux may be determined by noting the direction of the compass pointer. The direction of a magnetic field at any point is taken as that in which the north-seeking pole of a compass needle points when suspended in the field. The direction of a line of flux is from the north pole to the south pole on the outside of the magnet and is then assumed to continue through the magnet back to the point at which it emerged at the north pole. Thus such lines of flux always form complete closed loops or paths, they never intersect and always have a definite direction. The laws of magnetic attraction and repulsion can be demonstrated by using two bar magnets. In Figure 7.2(a), **with unlike poles adjacent, attraction takes place**. Lines of flux are imagined to contract and the magnets try to pull together. The magnetic field is strongest in between the two magnets, shown by the lines of flux being close together. In Figure 7.2(b), **with similar poles adjacent (i.e. two north poles), repulsion occurs**, i.e. the two north poles try to push each other apart, since magnetic flux lines running side by side in the same direction repel.

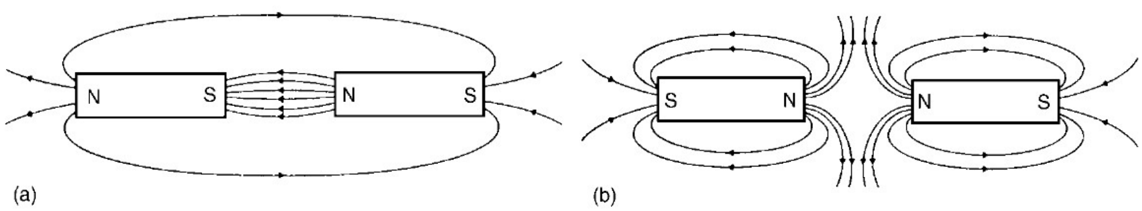


Figure 7.2

7.2 Magnetic flux and flux density

Magnetic flux is the amount of magnetic field (or the number of lines of force) produced by a magnetic source. The symbol for magnetic flux is Φ (Greek letter 'phi'). The unit of magnetic flux is the **weber, Wb**

Magnetic flux density is the amount of flux passing through a defined area that is perpendicular to the direction of the flux:

$$\textbf{Magnetic flux density} = \frac{\textbf{magnetic flux}}{\textbf{area}}$$

The symbol for magnetic flux density is B. The unit of magnetic flux density is the **tesla, T**, where $1\ \text{T} = 1\ \text{Wb/m}^2$ Hence

$$\boxed{\boldsymbol{B = \frac{\Phi}{A}} \textbf{ tesla}}, \text{ where } A(\text{m}^2) \text{ is the area}$$

Problem 1. A magnetic pole face has a rectangular section having dimensions 200 mm by 100 mm. If the total flux emerging from the pole is 150 μWb, calculate the flux density.

Flux $\Phi = 150\ \mu\text{Wb} = 150 \times 10^{-6}$ Wb

Cross sectional area $A = 200 \times 100 = 20000\ \text{mm}^2$

$= 20000 \times 10^{-6}\ \text{m}^2$

$$\text{Flux density } B = \frac{\Phi}{A} = \frac{150 \times 10^{-6}}{20000 \times 10^{-6}}$$

$= \mathbf{0.0075\ T\ or\ 7.5\ mT}$

Problem 2. The maximum working flux density of a lifting electromagnet is 1.8 T and the effective area of a pole face is circular in cross-section. If the total magnetic flux produced is 353 mWb, determine the radius of the pole face.

Flux density $B = 1.8$ T; flux $\Phi = 353\ \text{mWb} = 353 \times 10^{-3}$ Wb

$$\text{Since } B = \frac{\Phi}{A}, \text{ cross-sectional area } A = \frac{\Phi}{B} = \frac{353 \times 10^{-3}}{1.8}\ \text{m}^2$$

$= 0.1961\ \text{m}^2$

The pole face is circular, hence area $= \pi r^2$, where r is the radius.

Hence $\pi r^2 = 0.1961$

from which $r^2 = \dfrac{0.1961}{\pi}$ and radius $r = \sqrt{\left(\dfrac{0.1961}{\pi}\right)} = 0.250$ m

i.e. **the radius of the pole face is 250 mm**

7.3 Magnetomotive force and magnetic field strength

Magnetomotive force (mmf) is the cause of the existence of a magnetic flux in a magnetic circuit,

$$\boxed{\textbf{mmf}, \boldsymbol{F_m = NI} \textbf{ amperes}}$$

where N is the number of conductors (or turns) and I is the current in amperes. The unit of mmf is sometimes expressed as 'ampere-turns'. However since 'turns' have no dimensions, the SI unit of mmf is the ampere. **Magnetic field strength** (or **magnetizing force**),

$$\boldsymbol{H = NI/l} \textbf{ ampere per metre,}$$

where l is the mean length of the flux path in metres.
Thus **mmf** $= \boldsymbol{NI} = \boldsymbol{Hl}$ **amperes**.

> Problem 3. A magnetizing force of 8000 A/m is applied to a circular magnetic circuit of mean diameter 30 cm by passing a current through a coil wound on the circuit. If the coil is uniformly wound around the circuit and has 750 turns, find the current in the coil.

$H = 8000$ A/m; $l = \pi d = \pi \times 30 \times 10^{-2}$ m; $N = 750$ turns

Since $H = \dfrac{NI}{l}$ then, $I = \dfrac{Hl}{N} = \dfrac{8000 \times \pi \times 30 \times 10^{-2}}{750}$

Thus, **current** $\boldsymbol{I = 10.05}$ **A**

7.4 Permeability and B–H curves

For air, or any non-magnetic medium, the ratio of magnetic flux density to magnetizing force is a constant, i.e. B/H = a constant. This constant is μ_0, the **permeability of free space** (or the magnetic space constant) and is equal to $4\pi \times 10^{-7}$ H/m, i.e., **for air, or any non-magnetic medium**, the ratio $\boxed{\boldsymbol{B/H = \mu_0}}$ (Although all non-magnetic materials, including air, exhibit slight magnetic properties, these can effectively be neglected.)

For all media other than free space, $\boxed{\boldsymbol{B/H = \mu_0\mu_r}}$

where u_r is the relative permeability, and is defined as

$$\boldsymbol{\mu_r = \dfrac{\textbf{flux density in material}}{\textbf{flux density in a vacuum}}}$$

μ_r varies with the type of magnetic material and, since it is a ratio of flux densities, it has no unit. From its definition, μ_r for a vacuum is 1. $\boldsymbol{\mu_0\mu_r = \mu}$, called the **absolute permeability**

By plotting measured values of flux density B against magnetic field strength H, a **magnetization curve** (or **B–H curve**) is produced. For non-magnetic materials this is a straight line. Typical curves for four magnetic materials are shown in Figure 7.3.

The **relative permeability** of a ferromagnetic material is proportional to the slope of the B–H curve and thus varies with the magnetic field strength. The approximate range of values of relative permeability μ_r for some common magnetic materials are:

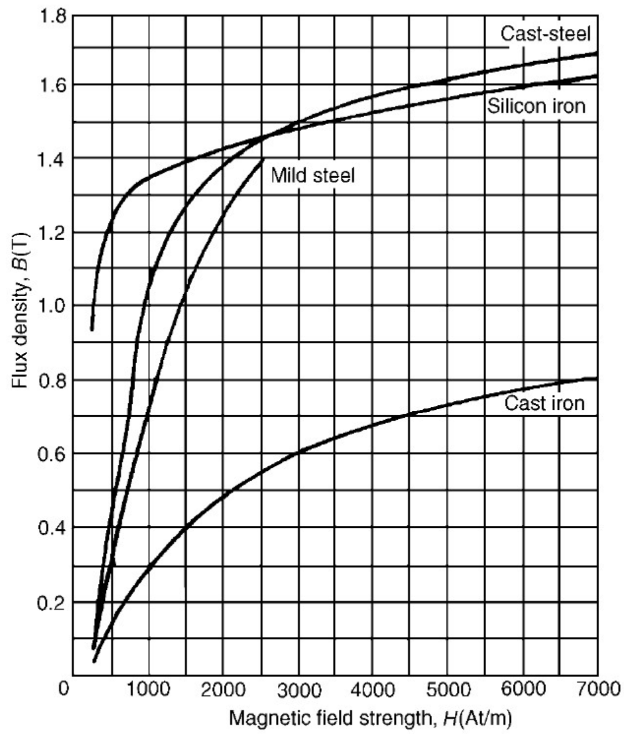


Figure 7.3 *B–H curves for four materials*

Cast iron	$\mu_r = 100–250$	Mild steel	$\mu_r = 200–800$
Silicon iron	$\mu_r = 1000–5000$	Cast steel	$\mu_r = 300–900$
Mumetal	$\mu_r = 200–5000$	Stalloy	$\mu_r = 500–6000$

> Problem 4. A flux density of 1.2 T is produced in a piece of cast steel by a magnetizing force of 1250 A/m. Find the relative permeability of the steel under these conditions.

For a magnetic material:

$$B = \mu_0 \mu_r H$$

i.e. $$\boldsymbol{u_r} = \frac{B}{\mu_0 H} = \frac{1.2}{(4\pi \times 10^{-7})(1250)} = \mathbf{764}$$

> Problem 5. Determine the magnetic field strength and the mmf required to produce a flux density of 0.25 T in an air gap of length 12 mm.

For air: $B = \mu_0 H$ (since $\mu_r = 1$)

Magnetic field strength $H = \dfrac{B}{\mu_0} = \dfrac{0.25}{4\pi \times 10^{-7}} =$ **198 940 A/m**

$$\textbf{mmf} = Hl = 198\,940 \times 12 \times 10^{-3} = \textbf{2387 A}$$

> Problem 6. A coil of 300 turns is wound uniformly on a ring of non-magnetic material. The ring has a mean circumference of 40 cm and a uniform cross sectional area of 4 cm^2. If the current in the coil is 5 A, calculate (a) the magnetic field strength, (b) the flux density and (c) the total magnetic flux in the ring.

(a) Magnetic field strength $H = \dfrac{NI}{l} = \dfrac{300 \times 5}{40 \times 10^{-2}} =$ **3750 A/m**

(b) For a non-magnetic material $\mu_r = 1$, thus flux density $B = \mu_0 H$

i.e. $\mathbf{B} = 4\pi \times 10^{-7} \times 3750 =$ **4.712 mT**

(c) Flux $\Phi = BA = (4.712 \times 10^{-3})(4 \times 10^{-4}) =$ **1.885 μWb**

> Problem 7. An iron ring of mean diameter 10 cm is uniformly wound with 2000 turns of wire. When a current of 0.25 A is passed through the coil a flux density of 0.4 T is set up in the iron. Find (a) the magnetizing force and (b) the relative permeability of the iron under these conditions.

$l = \pi d = \pi \times 10$ cm $= \pi \times 10 \times 10^{-2}$ m; $N = 2000$ turns; $I = 0.25$ A; $B = 0.4$ T

(a) $\mathbf{H} = \dfrac{NI}{l} = \dfrac{2000 \times 0.25}{\pi \times 10 \times 10^{-2}} = \dfrac{5000}{\pi} =$ **1592 A/m**

(b) $B = \mu_0 \mu_r H$, hence $\boldsymbol{\mu_r} = \dfrac{B}{\mu_0 H} = \dfrac{0.4}{(4\pi \times 10^{-7})(1592)} =$ **200**

> Problem 8. A uniform ring of cast iron has a cross-sectional area of 10 cm^2 and a mean circumference of 20 cm. Determine the mmf necessary to produce a flux of 0.3 mWb in the ring. The magnetization curve for cast iron is shown on page 78.

$A = 10$ cm$^2 = 10 \times 10^{-4}$ m^2; $l = 20$ cm $= 0.2$ m; $\Phi = 0.3 \times 10^{-3}$ Wb

Flux density $B = \dfrac{\Phi}{A} = \dfrac{0.3 \times 10^{-3}}{10 \times 10^{-4}} = 0.3$ T

From the magnetization curve for cast iron on page 78, when $B = 0.3$ T, $H = 1000$ A/m, hence mmf $= Hl = 1000 \times 0.2 =$ **200 A**

A tabular method could have been used in this problem. Such a solution is shown below.

Part of circuit	Material	Φ (Wb)	A (m^2)	$B = \dfrac{\Phi}{A}$ (T)	H from graph	l (m)	mmf $= Hl$ (A)
Ring	Cast iron	0.3×10^{-3}	10×10^{-4}	0.3	1000	0.2	200

7.5 Reluctance

Reluctance S (or R_M) is the 'magnetic resistance' of a magnetic circuit to the presence of magnetic flux.

$$\textbf{Reluctance } S = \frac{F_M}{\Phi} = \frac{NI}{\Phi} = \frac{Hl}{BA} = \frac{l}{(B/H)A} = \frac{\boldsymbol{l}}{\boldsymbol{\mu_0 \mu_r A}}$$

The unit of reluctance is 1/H (or H^{-1}) or A/Wb
Ferromagnetic materials have a low reluctance and can be used as **magnetic screens** to prevent magnetic fields affecting materials within the screen.

> Problem 9. Determine the reluctance of a piece of mumetal of length 150 mm and cross-sectional area 1800 mm^2 when the relative permeability is 4000. Find also the absolute permeability of the mumetal.

$$\textbf{Reluctance } S = \frac{l}{\mu_0 \mu_r A} = \frac{150 \times 10^{-3}}{(4\pi \times 10^{-7})(4000)(1800 \times 10^{-6})}$$
$$= \mathbf{16\ 580/H}$$

$$\textbf{Absolute permeability, } \boldsymbol{\mu} = \mu_0 \mu_r = (4\pi \times 10^{-7})(4000)$$
$$= \mathbf{5.027 \times 10^{-3}\ H/m}$$

> Problem 10. A mild steel ring has a radius of 50 mm and a cross-sectional area of 400 mm^2. A current of 0.5 A flows in a coil wound uniformly around the ring and the flux produced is 0.1 mWb. If the relative permeability at this value of current is 200 find (a) the reluctance of the mild steel and (b) the number of turns on the coil.

$l = 2\pi r = 2 \times \pi \times 50 \times 10^{-3}$ m; $A = 400 \times 10^{-6}$ m^2; $I = 0.5$ A; $\Phi = 0.1 \times 10^{-3}$ Wb; $\mu_r = 200$

(a) $$\textbf{Reluctance } S = \frac{l}{\mu_0 \mu_r A} = \frac{2 \times \pi \times 50 \times 10^{-3}}{(4\pi \times 10^{-7})(200)(400 \times 10^{-6})}$$
$$= \mathbf{3.125 \times 10^6/H}$$

(b) $S = \dfrac{\text{mmf}}{\Phi}$ i.e. $\text{mmf} = S\Phi$

so that $NI = S\Phi$ and

hence $N = \dfrac{S\Phi}{I} = \dfrac{3.125 \times 10^6 \times 0.1 \times 10^{-3}}{0.5} = \mathbf{625\ turns}$

Further problems on magnetic circuit quantities may be found in Section 7.9, problems 1 to 14, page 85.

7.6 Composite series magnetic circuits

For a series magnetic circuit having n parts, the **total reluctance S** is given by:

$$\boldsymbol{S = S_1 + S_2 + \ldots + S_n}$$

(This is similar to resistors connected in series in an electrical circuit.)

> Problem 11. A closed magnetic circuit of cast steel contains a 6 cm long path of cross-sectional area 1 cm^2 and a 2 cm path of cross-sectional area 0.5 cm^2. A coil of 200 turns is wound around the 6 cm length of the circuit and a current of 0.4 A flows. Determine the flux density in the 2 cm path, if the relative permeability of the cast steel is 750.

For the 6 cm long path:

$$\text{Reluctance } S_1 = \frac{l_1}{\mu_0\mu_r A_1} = \frac{6 \times 10^{-2}}{(4\pi \times 10^{-7})(750)(1 \times 10^{-4})}$$

$$= 6.366 \times 10^5/\text{H}$$

For the 2 cm long path:

$$\text{Reluctance } S_2 = \frac{l_2}{\mu_0\mu_r A_2} = \frac{2 \times 10^{-2}}{(4\pi \times 10^{-7})(750)(0.5 \times 10^{-4})}$$

$$= 4.244 \times 10^5/\text{H}$$

$$\text{Total circuit reluctance } S = S_1 + S_2 = (6.366 + 4.244) \times 10^5$$

$$= 10.61 \times 10^5/\text{H}$$

$$S = \frac{\text{mmf}}{\Phi}, \text{ i.e. } \Phi = \frac{\text{mmf}}{S} = \frac{NI}{S} = \frac{200 \times 0.4}{10.61 \times 10^5} = 7.54 \times 10^{-5}\ \text{Wb}$$

$$\text{Flux density in the 2 cm path, } B = \frac{\Phi}{A} = \frac{7.54 \times 10^{-5}}{0.5 \times 10^{-4}} = \mathbf{1.51\ T}$$

Problem 12. A silicon iron ring of cross-sectional area 5 cm^2 has a radial air gap of 2 mm cut into it. If the mean length of the silicon iron path is 40 cm, calculate the magnetomotive force to produce a flux of 0.7 mWb. The magnetization curve for silicon is shown on page 78.

There are two parts to the circuit — the silicon iron and the air gap. The total mmf will be the sum of the mmf's of each part.

For the silicon iron: $B = \dfrac{\Phi}{A} = \dfrac{0.7 \times 10^{-3}}{5 \times 10^{-4}} = 1.4$ T

From the B–H curve for silicon iron on page 78, when $B = 1.4$ T, $H = 1650$ At/m.

Hence the mmf for the iron path $= Hl = 1650 \times 0.4 = 660$ A

For the air gap:

The flux density will be the same in the air gap as in the iron, i.e. 1.4 T. (This assumes no leakage or fringing occurring.)

$$\text{For air, } H = \frac{B}{\mu_0} = \frac{1.4}{4\pi \times 10^{-7}} = 1\,114\,000 \text{ A/m}$$

$$\text{Hence the mmf for the air gap} = Hl = 1\,114\,000 \times 2 \times 10^{-3} = 2228 \text{ A}$$

Total mmf to produce a flux of 0.7 mWb $= 660 + 2228$

$= \mathbf{2888\ A}$

A tabular method could have been used as shown below.

Part of circuit	Material	Φ (Wb)	A (m^2)	B (T)	H (A/m)	l (m)	mmf = Hl (A)
Ring	Silicon iron	0.7×10^{-3}	5×10^{-4}	1.4	1650 (from graph)	0.4	660
Air-gap	Air	0.7×10^{-3}	5×10^{-4}	1.4	$\dfrac{1.4}{4\pi \times 10^{-7}} = 1\,114\,000$	2×10^{-3}	2228
						Total:	**2888 A**

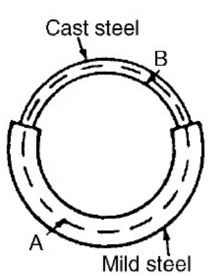


Figure 7.4

Problem 13. Figure 7.4 shows a ring formed with two different materials — cast steel and mild steel. The dimensions are:

	mean length	cross-sectional area
Mild steel	400 mm	500 mm^2
Cast steel	300 mm	312.5 mm^2

Find the total mmf required to cause a flux of 500 μWb in the magnetic circuit. Determine also the total circuit reluctance.

A tabular solution is shown below.

Part of circuit	Material	Φ (Wb)	A (m^2)	B (T) ($= \Phi/A$)	H (A/m) (from graphs p 78)	l (m)	mmf $= Hl$ (A)
A	Mild steel	500×10^{-6}	500×10^{-6}	1.0	1400	400×10^{-3}	560
B	Cast steel	500×10^{-6}	312.5×10^{-6}	1.6	4800	300×10^{-3}	1440
							Total: **2000 A**

Total circuit reluctance $S = \dfrac{\text{mmf}}{\Phi} = \dfrac{2000}{500 \times 10^{-6}}$

$= \mathbf{4 \times 10^6/H}$

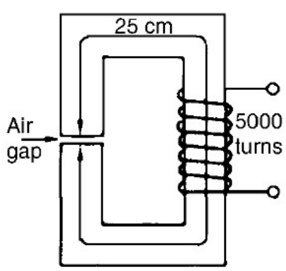


Figure 7.5

Problem 14. A section through a magnetic circuit of uniform cross-sectional area 2 cm^2 is shown in Figure 7.5. The cast steel core has a mean length of 25 cm. The air gap is 1 mm wide and the coil has 5000 turns. The B–H curve for cast steel is shown on page 78. Determine the current in the coil to produce a flux density of 0.80 T in the air gap, assuming that all the flux passes through both parts of the magnetic circuit.

For the cast steel core, when $B = 0.80$ T, $H = 750$ A/m (from page 78)

Reluctance of core $S_1 = \dfrac{l_1}{\mu_0 \mu_r A_1}$ and since $B = \mu_0 \mu_r H$,

then $\mu_r = \dfrac{B}{\mu_0 H}$. Thus $S_1 = \dfrac{l_1}{\mu_0 \left(\dfrac{B}{\mu_0 H}\right) A} = \dfrac{l_1 H}{BA} = \dfrac{(25 \times 10^{-2})(750)}{(0.8)(2 \times 10^{-4})}$

$= 1\,172\,000/\text{H}$

For the air gap: Reluctance, $S_2 = \dfrac{l_2}{\mu_0 \mu_r A_2} = \dfrac{l_2}{\mu_0 A_2}$

(since $\mu_r = 1$ for air)

$= \dfrac{1 \times 10^{-3}}{(4\pi \times 10^{-7})(2 \times 10^{-4})}$

$= 3\,979\,000/\text{H}$

Total circuit reluctance $S = S_1 + S_2 = 1\,172\,000 + 3\,979\,000$

$= 5\,151\,000/\text{H}$

Flux $\Phi = BA = 0.80 \times 2 \times 10^{-4} = 1.6 \times 10^{-4}$ Wb

$$S = \frac{\text{mmf}}{\Phi}, \text{ thus mmf} = S\Phi$$

Hence $NI = S\Phi$

and current $I = \frac{S\Phi}{N} = \frac{(5\,151\,000)(1.6 \times 10^{-4})}{5000} = \mathbf{0.165\ A}$

Further problems on composite series magnetic circuits may be found in Section 7.9, problems 15 to 19, page 86.

7.7 Comparison between electrical and magnetic quantities

Electrical circuit		Magnetic circuit	
e.m.f. E	(V)	mmf F_m	(A)
current I	(A)	flux Φ	(Wb)
resistance R	(Ω)	reluctance S	(H^{-1})
$I = \frac{E}{R}$		$\Phi = \frac{\text{mmf}}{S}$	
$R = \frac{\rho l}{A}$		$S = \frac{l}{\mu_0 \mu_r A}$	

7.8 Hysteresis and hysteresis loss

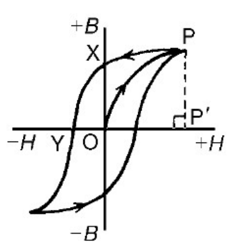


Figure 7.6

Hysteresis is the 'lagging' effect of flux density B whenever there are changes in the magnetic field strength H. When an initially unmagnetized ferromagnetic material is subjected to a varying magnetic field strength H, the flux density B produced in the material varies as shown in Figure 7.6, the arrows indicating the direction of the cycle. Figure 7.6 is known as a **hysteresis loop**.

From Figure 7.6, distance OX indicates the **residual flux density** or **remanence**, OY indicates the **coercive force**, and PP' is the **saturation flux density**.

Hysteresis results in a dissipation of energy which appears as a heating of the magnetic material. **The energy loss associated with hysteresis is proportional to the area of the hysteresis loop**.

The production of the hysteresis loop and hysteresis loss are explained in greater detail in Chapter 38, Section 3, page 692.

The area of a hysteresis loop varies with the type of material. The area, and thus the energy loss, is much greater for hard materials than for soft materials.

For AC-excited devices the hysteresis loop is repeated every cycle of alternating current. Thus a hysteresis loop with a large area (as with hard steel) is often unsuitable since the energy loss would be considerable. Silicon steel has a narrow hysteresis loop, and thus small hysteresis loss, and is suitable for transformer cores and rotating machine armatures.

3

Analysis of AC Circuits

3.1 INTRODUCTION

We have discussed the network theorems with reference to resistive load and dc sources. Now, all the theorems will be discussed when a network consists of ac sources, resistors, inductors and capacitors. All the theorems are also valid for ac sources.

3.2 MESH ANALYSIS

Mesh analysis is useful if a network has a large number of voltage sources. In this method, currents are assigned in each mesh. We can write mesh equations by Kirchhoff's voltage law in terms of unknown mesh currents,

Example 3.1 *Find mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the network of Fig. 3.1.*

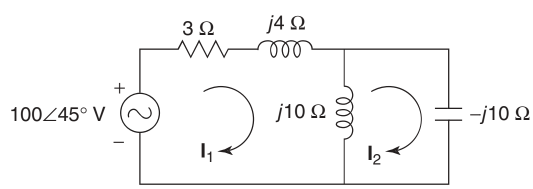


Fig. 3.1

Solution Applying KVL to Mesh 1,

$$100\angle 45^\circ - (3 + j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(3 + j14)\mathbf{I}_1 - j10\mathbf{I}_2 = 100\angle 45^\circ \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) + j10(\mathbf{I}_2) = 0$$

$$j10\mathbf{I}_1 = 0 \quad \text{...(ii)}$$

$$\mathbf{I}_1 = 0$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$-j10\mathbf{I}_2 = 100\angle 45^\circ$$

$$\mathbf{I}_2 = \frac{100\angle 45^\circ}{-j10} = 10\angle 135^\circ \text{ A}$$

Example 3.2 *Find mesh current I_1, I_2 and I_3 in the network of Fig. 3.2.*

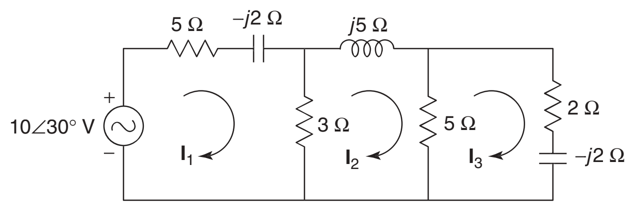


Fig. 3.2

Solution Applying KVL to Mesh 1,

$$10\angle 30^\circ - (5 - j2)\,\mathbf{I}_1 - 3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(8 - j2)\,\mathbf{I}_1 - 3\mathbf{I}_2 = 10\angle 30^\circ \qquad \dots \text{(i)}$$

Applying KVL to Mesh 2,

$$-3(\mathbf{I}_2 - \mathbf{I}_1) - j5\mathbf{I}_2 - 5(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-3\mathbf{I}_1 + (8 + j5)\,\mathbf{I}_2 - 5\mathbf{I}_3 = 0 \qquad \dots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-5(\mathbf{I}_3 - \mathbf{I}_2) - (2 - j2)\,\mathbf{I}_3 = 0$$

$$-5\mathbf{I}_2 + (7 - j2)\,\mathbf{I}_3 = 0 \qquad \dots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 30^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 30^\circ & -3 & 0 \\ 0 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 1.43\angle 38.7^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 8 - j2 & 10\angle 30^\circ & 0 \\ -3 & 0 & -5 \\ 0 & 0 & 7 - j2 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 0.693\angle -2.2^\circ \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 8 - j2 & -3 & 10\angle 30^\circ \\ -3 & 8 + j5 & 0 \\ 0 & -5 & 0 \end{vmatrix}}{\begin{vmatrix} 8 - j2 & -3 & 0 \\ -3 & 8 + j5 & -5 \\ 0 & -5 & 7 - j2 \end{vmatrix}} = 0.476\angle 13.8^\circ \text{ A}$$

Example 3.3 *In the network of Fig. 3.3, find the value of $\mathbf{V}_2$ so that the current through (2 + j3) ohm impedance is zero.*

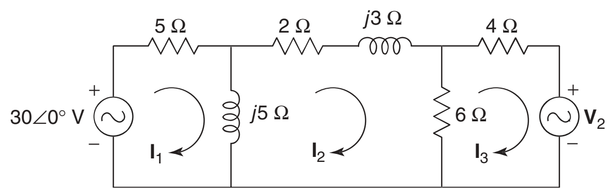


Fig. 3.3

Solution Applying KVL to Mesh 1,

$$30\angle 0° - 5\mathbf{I}_1 - j5(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(5 + j5)\mathbf{I}_1 - j5\mathbf{I}_2 = 30\angle 0° \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-j5(\mathbf{I}_2 - \mathbf{I}_1) - (2 + j3)\mathbf{I}_2 - 6(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j5\mathbf{I}_1 + (8 + j8)\mathbf{I}_2 - 6\mathbf{I}_3 = 0 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-6(\mathbf{I}_3 - \mathbf{I}_2) - 4\mathbf{I}_3 - \mathbf{V}_2 = 0$$

$$-6\mathbf{I}_2 + 10\mathbf{I}_3 = -\mathbf{V}_2 \qquad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 30\angle 0° \\ 0 \\ -\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 5+j5 & 30\angle 0° & 0 \\ -j5 & 0 & -6 \\ 0 & -\mathbf{V}_2 & 10 \end{vmatrix}}{\begin{vmatrix} 5+j5 & -j5 & 0 \\ -j5 & 8+j8 & -6 \\ 0 & -6 & 10 \end{vmatrix}} = 0$$

$$(5 + j5)(-6\mathbf{V}_2) - (30)(-j50) = 0$$

$$\mathbf{V}_2 = \frac{j1500}{30 + j30} = 35.36\angle 45° \text{ V}$$

Example 3.4 *Find the value of the current $\mathbf{I}_3$ in the network shown in Fig. 3.4.*

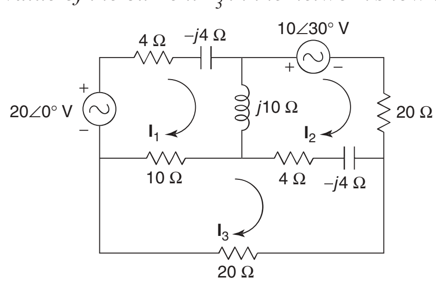


Fig. 3.4

Solution Applying KVL to Mesh 1,

$$20\angle 0° - (4 - j4)\mathbf{I}_1 - j10(\mathbf{I}_1 - \mathbf{I}_2) - 10(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(14 + j6)\mathbf{I}_1 - j10\mathbf{I}_2 - 10\mathbf{I}_3 = 20\angle 0° \qquad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-j10(\mathbf{I}_2 - \mathbf{I}_1) - 10\angle 30° - 20\mathbf{I}_2 - (4 - j4)(\mathbf{I}_2 - \mathbf{I}_3) = 0$$

$$-j10\mathbf{I}_1 + (24 + j6)\mathbf{I}_2 - (4 - j4)\mathbf{I}_3 = -10\angle 30° \qquad \ldots \text{(ii)}$$

Applying KVL to Mesh 3,

$$-10(\mathbf{I}_3 - \mathbf{I}_1) - (4 - j4)(\mathbf{I}_3 - \mathbf{I}_2) - 20\mathbf{I}_3 = 0$$

$$-10\mathbf{I}_1 - (4 - j4)\mathbf{I}_2 + (34 - j4)\mathbf{I}_3 = 0 \qquad \ldots \text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 14 + j6 & -j10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 20\angle 0° \\ -10\angle 30° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 14 + j6 & -j10 & 20\angle 0° \\ -j10 & 24 + j6 & -10\angle 30° \\ -10 & -(4 - j4) & 0 \end{vmatrix}}{\begin{vmatrix} 14 + j6 & -10 & -10 \\ -j10 & 24 + j6 & -(4 - j4) \\ -10 & -(4 - j4) & 34 - j4 \end{vmatrix}} = 0.44\angle -14° \text{ A}$$

Example 3.5 *Find the voltage V_{AB} in the network of Fig. 3.5.*

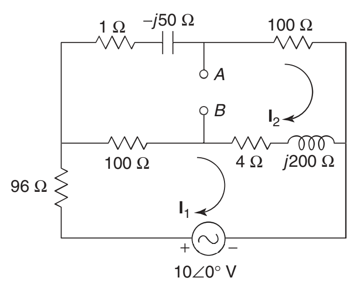


Fig. 3.5

Solution Applying KVL to Mesh 1,

$$-96\,\mathbf{I}_1 - (100 + 4 + j200)(\mathbf{I}_1 - \mathbf{I}_2) + 10\angle 0° = 0$$

$$(200 + j200)\,\mathbf{I}_1 - (104 + j200)\,\mathbf{I}_2 = 10\angle 0° \qquad \ldots \text{(i)}$$

Applying KVL to Mesh 2,

$$-(1 - j50 - 100)\,\mathbf{I}_2 - (100 + 4 + j200)(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$-(104 + j200)\,\mathbf{I}_1 + (205 + j150)\,\mathbf{I}_2 = 0 \qquad \ldots \text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 200 + j200 & -(104 + j200) \\ -(104 + j200) & 205 + j150 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -(104+j200) \\ 0 & 205+j150 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.051\angle 2.72\times 10^{-3}\,{}^\circ\text{A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 200+j200 & 10\angle 0^\circ \\ -(104+j200) & 0 \end{vmatrix}}{\begin{vmatrix} 200+j200 & -(104+j200) \\ -(104+j200) & 205+j150 \end{vmatrix}} = 0.045\angle 26.34^\circ \text{ A}$$

$$\begin{aligned} \mathbf{V}_{AB} &= 100\mathbf{I}_2 - (4+j200)(\mathbf{I}_1 - \mathbf{I}_2) \\ &= 100(0.045\angle 26.34^\circ) - (4+j200)(0.051\angle 2.72\times 10^{-3}\,{}^\circ - 0.045\angle 26.34^\circ) \\ &= 0.058\angle -92.65^\circ \text{ V} \end{aligned}$$

Example 3.6 *For the network shown in Fig. 3.6, find the voltage across the capacitor.*

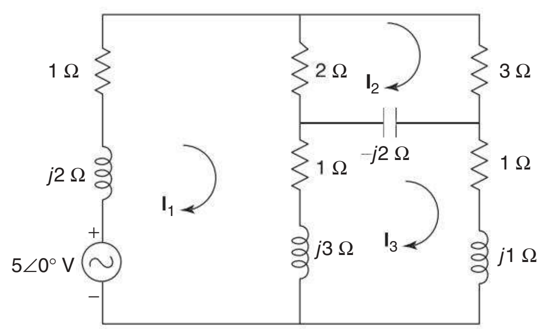


Fig. 3.6

Solution Applying KVL to Mesh 1,

$$5\angle 0^\circ - (1+j2)\mathbf{I}_1 - 2(\mathbf{I}_1 - \mathbf{I}_2) - (1+j3)(\mathbf{I}_1 - \mathbf{I}_3) = 0$$

$$(4+j5)\mathbf{I}_1 - 2\mathbf{I}_2 - (1+j3)\mathbf{I}_3 = 5\angle 0^\circ \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-2(\mathbf{I}_2 - \mathbf{I}_1) - 3\mathbf{I}_2 + j2(\mathbf{I}_2 - \mathrm{I}_3) = 0$$

$$-2\mathbf{I}_1 + (5-j2)\mathbf{I}_2 + j2\mathbf{I}_3 = 0 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-(1+j3)(\mathbf{I}_3 - \mathbf{I}_1) + j2(\mathbf{I}_3 - \mathrm{I}_2) - (1+j1)\mathbf{I}_3 = 0$$

$$-(1+j3)\mathbf{I}_1 + j2\mathbf{I}_2 + (2+j2)\mathbf{I}_3 = 0 \quad \text{...(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 5\angle 0^\circ \\ 0 \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4+j5 & 5\angle 0° & -(1+j3) \\ -2 & 0 & j2 \\ -(1+j3) & 0 & 2+j2 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.65\angle 130.51° \text{ A}$$

$$\mathbf{I}_3 = \frac{\begin{vmatrix} 4+j5 & -2 & 5\angle 0° \\ -2 & 5-j2 & 0 \\ -(1+j3) & j2 & 0 \end{vmatrix}}{\begin{vmatrix} 4+j5 & -2 & -(1+j3) \\ -2 & 5-j2 & j2 \\ -(1+j3) & j2 & 2+j2 \end{vmatrix}} = 0.91\angle -21.51° \text{ A}$$

$$\mathbf{V}_c = (-j2)(\mathbf{I}_3 - \mathbf{I}_2) = (-j2)(0.91\angle -21.51° - 0.65\angle 130.51°) = 3.03\angle -123.12° \text{ V}$$

Example 3.7 *Find the voltage across the 2 Ω resistor in the network of Fig. 3.7.*

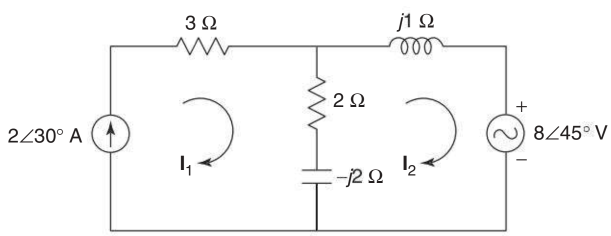


Fig. 3.7

Solution For Mesh 1,

$$\mathbf{I}_1 = 2\angle 30° \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$-(2-j2)(\mathbf{I}_2 - \mathbf{I}_1) - j1\mathbf{I}_2 - 8\angle 45° = 0$$

$$(2-j2)\mathbf{I}_1 - (2-j1)\mathbf{I}_2 = 8\angle 45° \quad \text{...(ii)}$$

Substituting $\mathbf{I}_1$ in Eq. (i),

$$(2-j2)(2\angle 30°) - (2-j1)\mathbf{I}_2 = 8\angle 45°$$

$$\mathbf{I}_2 = \frac{-(8\angle 45°) + (2-j2)(2\angle 30°)}{2-j1} = 3.19\angle -65° \text{ A}$$

$$\mathbf{V}_{2\,\Omega} = 2(\mathbf{I}_1 - \mathbf{I}_2) = 2(2\angle 30° - 3.19\angle -65°) = 7.82\angle 84.37° \text{ V}$$

Example 3.8 *Find the current through 3 Ω resistor in the network of Fig. 3.8.*

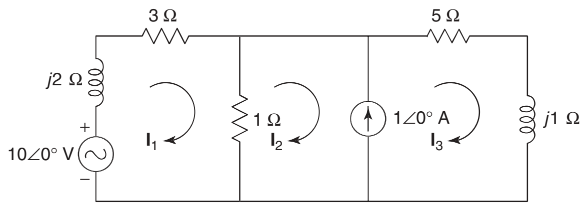


Fig. 3.8

Solution Applying KVL to Mesh 1,

$$10\angle 0^\circ - j2\mathbf{I}_1 - 3\mathbf{I}_1 - 1(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(4 + j2)\mathbf{I}_1 - \mathbf{I}_2 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Meshes 2 and 3 will form a supermesh.
Writing current equation for the supermesh,

$$\mathbf{I}_3 - \mathbf{I}_2 = 1\angle 0^\circ \qquad \ldots\text{(ii)}$$

Applying KVL to the outer path of the supermesh,

$$-1(\mathbf{I}_2 - \mathbf{I}_1) - 5\mathbf{I}_3 - j1\mathbf{I}_3 = 0$$

$$\mathbf{I}_1 - \mathbf{I}_2 - (5 + j1)\mathbf{I}_3 = 0 \qquad \ldots\text{(iii)}$$

Writing Eqs (i), (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 1\angle 0^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -1 & 0 \\ 1\angle 0^\circ & -1 & 1 \\ 0 & -1 & -(5+j1) \end{vmatrix}}{\begin{vmatrix} 4+j2 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -(5+j1) \end{vmatrix}} = 2.11\angle -28.01^\circ \text{ A}$$

$$\mathbf{I}_{3\,\Omega} = \mathbf{I}_1 = 2.11\angle -28.01^\circ \text{ A}$$

Example 3.9 *Find the currents $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ in the network of Fig. 3.9.*

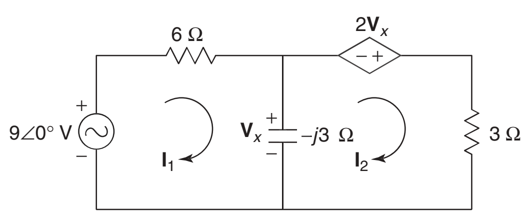


Fig. 3.9

Solution From Fig. 3.9,

$$\mathbf{V}_x = -j3(\mathbf{I}_1 - \mathbf{I}_2) \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$9\angle 0° - 6\mathbf{I}_1 + j3(\mathbf{I}_1 - \mathbf{I}_2) = 0$$

$$(6 - j3)\mathbf{I}_1 + j3\mathbf{I}_2 = 9\angle 0° \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$j3(\mathbf{I}_2 - \mathbf{I}_1) + 2\mathbf{V}_x - 3\mathbf{I}_2 = 0$$

$$j3\mathbf{I}_2 - j3\mathbf{I}_1 + 2[-j3(\mathbf{I}_1 - \mathbf{I}_2)] - 3\mathbf{I}_2 = 0$$

$$j9\mathbf{I}_1 + (3 - j9)\mathbf{I}_2 = 0 \quad \text{...(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 6-j3 & j3 \\ j9 & 3-j9 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 9\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 9\angle 0° & j3 \\ 0 & 3-j9 \end{vmatrix}}{\begin{vmatrix} 6-j3 & j3 \\ j9 & 3-j9 \end{vmatrix}} = 1.3\angle 2.49° \text{ A}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 6-j3 & 9\angle 0° \\ j9 & 0 \end{vmatrix}}{\begin{vmatrix} 6-j3 & j3 \\ j9 & 3-j9 \end{vmatrix}} = 1.24\angle -15.95° \text{ A}$$

Example 3.10 *Find the voltage across the 4 Ω resistor in the network of Fig. 3.10.*

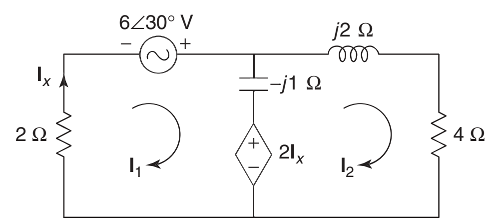


Fig. 3.10

Solution From Fig. 3.10,

$$\mathbf{I}_x = \mathbf{I}_1 \quad \text{...(i)}$$

Applying KVL to Mesh 1,

$$-2\mathbf{I}_1 + 6\angle 30° + j1(\mathbf{I}_1 - \mathbf{I}_2) - 2\mathbf{I}_x = 0$$

$$-2\mathbf{I}_1 + 6\angle 30° + j1\mathbf{I}_1 - j1\mathbf{I}_2 - 2\mathbf{I}_1 = 0$$

$$(4 - j1)\mathbf{I}_1 + j1\mathbf{I}_2 = 6\angle 30° \quad \text{...(ii)}$$

Applying KVL to Mesh 2,

$$2\mathbf{I}_x + j1(\mathbf{I}_2 - \mathbf{I}_1) - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$2\mathbf{I}_1 + j1\mathbf{I}_2 - j1\,\mathbf{I}_1 - j2\mathbf{I}_2 - 4\mathbf{I}_2 = 0$$

$$(2 - j1)\,\mathbf{I}_1 - (4 + j1)\,\mathbf{I}_2 = 0 \qquad \text{...(iii)}$$

Writing Eqs (ii) and (iii) in matrix form,

$$\begin{bmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 6\angle 30^\circ \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{I}_2 = \frac{\begin{vmatrix} 4 - j1 & 6\angle 30^\circ \\ 2 - j1 & 0 \end{vmatrix}}{\begin{vmatrix} 4 - j1 & j1 \\ 2 - j1 & -(4 + j1) \end{vmatrix}} = 0.74\angle -2.91^\circ \text{ A}$$

$$V_{4\Omega} = 4\mathbf{I}_2 = 4(0.74\angle -2.91^\circ) = 2.96\angle -2.91^\circ \text{ V}$$

3.3 NODE ANALYSIS

Node analysis uses Kirchhoff's current law for finding currents and voltages in a network. For ac networks, Kirchhoff's current law states that the phasor sum of currents meeting at a point is equal to zero.

Example 3.11 *In the network shown in Fig. 3.11, determine $\mathbf{V}_a$ and $\mathbf{V}_b$.*

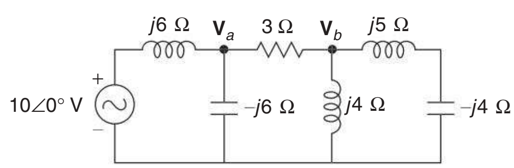


Fig. 3.11

Solution Applying KCL at Node *a*,

$$\frac{\mathbf{V}_a - 10\angle 0^\circ}{j6} + \frac{\mathbf{V}_a}{-j6} + \frac{\mathbf{V}_a - \mathbf{V}_b}{3} = 0$$

$$\left(\frac{1}{j6} - \frac{1}{j6} + \frac{1}{3}\right)\mathbf{V}_a - \frac{1}{3}\mathbf{V}_b = \frac{10\angle 0^\circ}{j6}$$

$$0.33\mathbf{V}_a - 0.33\mathbf{V}_b = 1.67\angle -90^\circ \qquad \text{...(i)}$$

Applying KCL at Node *b*,

$$\frac{\mathbf{V}_b - \mathbf{V}_a}{3} + \frac{\mathbf{V}_b}{j4} + \frac{\mathbf{V}_b}{j1} = 0$$

$$-\frac{1}{3}\mathbf{V}_a + \left(\frac{1}{3} + \frac{1}{j4} + \frac{1}{j1}\right)\mathbf{V}_b = 0$$

$$-0.33\mathbf{V}_a + (0.33 - j1.25)\mathbf{V}_b = 0 \qquad \text{...(ii)}$$

Adding Eqs (i) and (ii),

$$-j1.25\mathbf{V}_b = 1.67\angle-90°$$

$$\mathbf{V}_b = \frac{1.67\angle-90°}{-j1.25} = 1.34\angle 0°\text{ V}$$

Substituting $\mathbf{V}_b$ in Eq. (i),

$$0.33\mathbf{V}_a - 0.33(1.34\angle 0°) = 1.67\angle-90°$$

$$\mathbf{V}_a = \frac{1.73\angle 75.17°}{0.33} = 5.24\angle-75.17°\text{ V}$$

Example 3.12 *For the network shown in Fig. 3.12, find the voltages $\mathbf{V}_1$ and $\mathbf{V}_2$.*

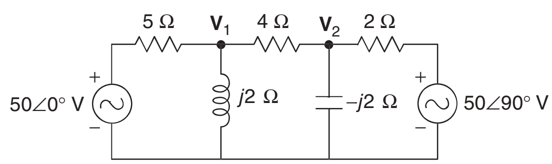


Fig. 3.12

Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle 0°}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_2 = 10\angle 0°$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_2}{-j2} + \frac{\mathbf{V}_2 - 50\angle 90°}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_2 = 25\angle 90°$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\mathbf{V}_2 = 25\angle 90° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 25\angle 90° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0° & -0.25 \\ j25 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 24.7\angle 72.25°\text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.45 - j0.5 & 10\angle 0° \\ -0.25 & 25\angle 90° \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = 34.34\angle 52.82°\text{ V}$$

Example 3.13 *Find the voltage V_{AB} in the network of Fig. 3.13.*

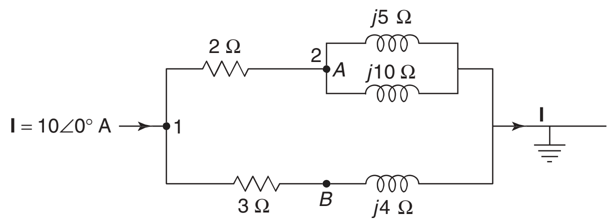


Fig. 3.13

Solution Applying KCL at Node 1,

$$10\angle 0° = \frac{\mathbf{V}_1 - \mathbf{V}_2}{2} + \frac{\mathbf{V}_1}{3+j4}$$

$$\left(\frac{1}{2} + \frac{1}{3+j4}\right)\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2 = 10\angle 0°$$

$$(0.62 - j0.16)\mathbf{V}_1 - 0.5\mathbf{V}_2 = 10\angle 0° \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{2} + \frac{\mathbf{V}_2}{j5} + \frac{\mathbf{V}_2}{j10} = 0$$

$$-\frac{1}{2}\mathbf{V}_1 + \left(\frac{1}{2} + \frac{1}{j5} + \frac{1}{j10}\right)\mathbf{V}_2 = 0$$

$$-0.5\mathbf{V}_1 + (0.5 - j0.3)\mathbf{V}_2 = 0 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0° \\ 0 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0° & -0.5 \\ 0 & 0.5 - j0.3 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 21.8\angle 56.42° \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.62 - j0.16 & 10\angle 0° \\ -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 0.62 - j0.16 & -0.5 \\ -0.5 & 0.5 - j0.3 \end{vmatrix}} = 18.7\angle 87.42° \text{ V}$$

$$\mathbf{V}_A = \mathbf{V}_2 = 18.7\angle 87.42° \text{ V}$$

$$\mathbf{V}_B = \frac{\mathbf{V}_1}{3+j4}(j4) = \frac{21.8\angle 56.42°}{3+j4}(j4) = 17.45\angle 93.32° \text{ V}$$

$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B = (18.7\angle 87.42°) - (17.45\angle 93.32°) = 2.23\angle 34.1° \text{ V}$$

Example 3.14 *Find the node voltages V_1 and V_2 in the network of Fig. 3.14.*

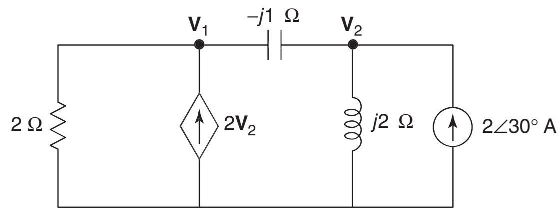


Fig. 3.14

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{-j1} = 2\mathbf{V}_2$$

$$\left(\frac{1}{2} + \frac{1}{-j1}\right)\mathbf{V}_1 - \left(2 - \frac{1}{j1}\right)\mathbf{V}_2 = 0$$

$$(0.5 + j1)\,\mathbf{V}_1 - (2 + j1)\mathbf{V}_2 = 0 \qquad \ldots\text{(i)}$$

Applying KCL at Node 2,

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{-j1} + \frac{\mathbf{V}_2}{j2} = 2\angle 30°$$

$$\frac{1}{j1}\mathbf{V}_1 + \left(\frac{1}{-j1} + \frac{1}{j2}\right)\mathbf{V}_2 = 2\angle 30°$$

$$-j1\,\mathbf{V}_1 + j\,0.5\,\mathbf{V}_2 = 2\angle 30° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.5 + j1 & -(2 + j1) \\ -j1 & j0.5 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\angle 30° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 0 & -(2 + j1) \\ 2\angle 30° & j0.5 \end{vmatrix}}{\begin{vmatrix} 0.5 + j1 & -(2 + j1) \\ -j1 & j0.5 \end{vmatrix}} = 2.46\angle 130.62° \text{ V}$$

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 0.5 + j1 & 0 \\ -j1 & 2\angle 30° \end{vmatrix}}{\begin{vmatrix} 0.5 + j1 & -(2 + j1) \\ -j1 & j0.5 \end{vmatrix}} = 1.23\angle 167.49° \text{ V}$$

Example 3.15 *In the network of Fig. 3.15, find the voltage V_2 which results in zero current through 4 Ω resistor.*

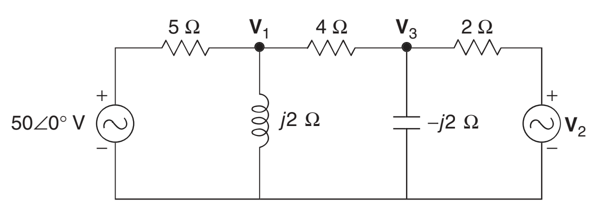


Fig. 3.15

Solution Applying KCL at Node 1,

$$\frac{\mathbf{V}_1 - 50\angle 0^\circ}{5} + \frac{\mathbf{V}_1}{j2} + \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\left(\frac{1}{5} + \frac{1}{j2} + \frac{1}{4}\right)\mathbf{V}_1 - \frac{1}{4}\mathbf{V}_3 = 10\angle 0^\circ$$

$$(0.45 - j0.5)\mathbf{V}_1 - 0.25\mathbf{V}_3 = 10\angle 0^\circ \qquad \ldots\text{(i)}$$

Applying KCL at Node 3,

$$\frac{\mathbf{V}_3 - \mathbf{V}_1}{4} + \frac{\mathbf{V}_3}{-j2} + \frac{\mathbf{V}_3 - \mathbf{V}_2}{2} = 0$$

$$-\frac{1}{4}\mathbf{V}_1 + \left(\frac{1}{4} + \frac{1}{-j2} + \frac{1}{2}\right)\mathbf{V}_3 = 0.5\mathbf{V}_2$$

$$-0.25\mathbf{V}_1 + (0.75 + j0.5)\,\mathbf{V}_3 = 0.5\mathbf{V}_2 \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{bmatrix}\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_3 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ 0.5\mathbf{V}_2 \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_1 = \frac{\begin{vmatrix} 10\angle 0^\circ & -0.25 \\ 0.5\mathbf{V}_2 & 0.75 + j0.5 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{10(0.75 + j0.5) + 0.125\,\mathbf{V}_2}{0.55\angle -15.95^\circ}$$

$$\mathbf{V}_3 = \frac{\begin{vmatrix} 0.45 - j\,0.5 & 10\angle 0^\circ \\ -0.25 & 0.5\mathbf{V}_2 \end{vmatrix}}{\begin{vmatrix} 0.45 - j0.5 & -0.25 \\ -0.25 & 0.75 + j0.5 \end{vmatrix}} = \frac{0.5\,\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$\mathbf{I}_{4\,\Omega} = \frac{\mathbf{V}_1 - \mathbf{V}_3}{4} = 0$$

$$\mathbf{V}_1 = \mathbf{V}_3$$

$$\frac{10(0.75 + j0.5) + 0.125\mathbf{V}_2}{0.55\angle -15.95^\circ} = \frac{0.5\mathbf{V}_2(0.45 - j0.5) + 2.5}{0.55\angle -15.95^\circ}$$

$$7.5 + 0.125\mathbf{V}_2 - j\,5 = 2.5 + 0.225\mathbf{V}_2 - j0.25\mathbf{V}_2$$

$$5 + j\,5 = \mathbf{V}_2(0.1 - j0.25)$$

$$\mathbf{V}_2 = \frac{5 + j5}{0.1 - j0.25} = 26.26\angle 113.2^\circ \text{ V}$$

Example 3.16 *Find the voltage across the capacitor in the network of Fig. 3.16.*

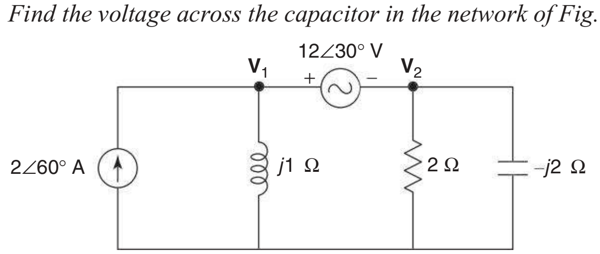


Fig. 3.16

Solution Nodes 1 and 2 will form a supernode.
Writing the voltage equation for the supernode,

$$\mathbf{V}_1 - \mathbf{V}_2 = 12\angle 30° \qquad \ldots\text{(i)}$$

Applying KCL to the supernode,

$$\frac{\mathbf{V}_1}{j1} + \frac{\mathbf{V}_2}{2} + \frac{\mathbf{V}_2}{-j2} = 2\angle 60°$$

$$(-j1)\,\mathbf{V}_1 + (0.5 + j0.5)\,\mathbf{V}_2 = 2\angle 60° \qquad \ldots\text{(ii)}$$

Writing Eqs (i) and (ii) in matrix form,

$$\begin{bmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 12\angle 30° \\ 2\angle 60° \end{bmatrix}$$

By Cramer's rule,

$$\mathbf{V}_2 = \frac{\begin{vmatrix} 1 & 12\angle 30° \\ -j1 & 2\angle 60° \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -j1 & 0.5 + j0.5 \end{vmatrix}} = 18.55\angle 157.42° \text{ V}$$

$$\mathbf{V}_c = \mathbf{V}_2 = 18.55\angle 157.42° \text{ V}$$

3.4 SUPERPOSITION THEOREM

The superposition theorem can be used to analyse an ac network containing more than one source. The superposition theorem states that *in a network containing more than one voltage source or current source, the total current or voltage in any branch of the network is the phasor sum of currents or voltages produced in that branch by each source acting separately*. As each source is considered, all of the other sources are replaced by their internal impedances. This theorem is valid only for linear systems.

Example 3.17 *Find the current through the 3 + j4 ohm impedance.*

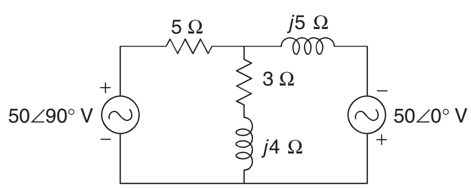


Fig. 3.17

Solution

Step I When the 50 ∠90° V source is acting alone (Fig. 3.18)

$$\mathbf{Z}_T = 5 + \frac{(3 + j4)(j5)}{3 + j9} = 6.35\angle 23.2°\ \Omega$$

$$\mathbf{I}_T = \frac{50\angle 90°}{6.35\angle 23.2°} = 7.87\angle 66.8° \text{ A}$$

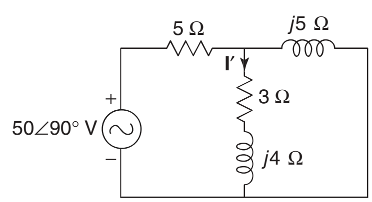


Fig. 3.18

2.7 STAR-DELTA TRANSFORMATION

When a circuit cannot be simplified by normal series–parallel reduction technique, the star-delta transformation can be used.

Figure 2.175(a) shows three resistors R_A, R_B and R_C connected in delta.

Figure 2.175(b) shows three resistors R_1, R_2 and R_3 connected in star.

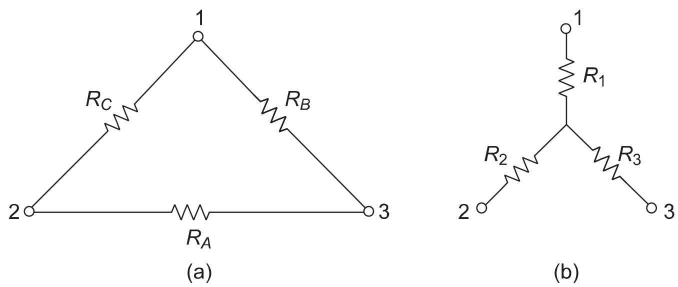


Fig. 2.175 *Delta and star networks*

These two networks will be electrically equivalent if the resistance as measured between any pair of terminals is the same in both the arrangements.

2.7.1 Delta to Star Transformation

Referring to delta network shown in Fig. 2.175(a),
the resistance between terminals 1 and 2 = $R_C \parallel (R_A + R_B)$

$$= \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \tag{2.4}$$

Referring to the star network shown in Fig. 2.175(b),
the resistance between terminals 1 and 2 = $R_1 + R_2$ (2.5)
Since the two networks are electrically equivalent,

$$R_1 + R_2 = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \tag{2.6}$$

Similarly, $$R_2 + R_3 = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} \tag{2.7}$$

and $$R_3 + R_1 = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} \tag{2.8}$$

Subtracting Eq. (2.7) from Eq. (2.6),

$$R_1 - R_3 = \frac{R_B R_C - R_A R_B}{R_A + R_B + R_C} \tag{2.9}$$

Adding Eq. (2.9) and Eq. (2.8),

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \tag{2.10}$$

Similarly,

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C} \tag{2.11}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \tag{2.12}$$

Thus, star resistor connected to a terminal is equal to the product of the two delta resistors connected to the same terminal divided by the sum of the delta resistors.

2.7.2 Star to Delta Transformation

Multiplying the above equations,

$$R_1\, R_2 = \frac{R_A R_B R_C^2}{(R_A + R_B + R_C)^2} \tag{2.13}$$

$$R_2\, R_3 = \frac{R_A^2 R_B R_C}{(R_A + R_B + R_C)^2} \tag{2.14}$$

$$R_3\, R_1 = \frac{R_A R_B^2 R_C}{(R_A + R_B + R_C)^2} \tag{2.15}$$

Adding Eqs (2.13), (2.14) and (2.15),

$$R_1\, R_2 + R_2\, R_3 + R_3\, R_1 = \frac{R_A R_B R_C (R_A + R_B + R_C)}{(R_A + R_B + R_C)^2}$$

$$= \frac{R_A R_B R_C}{R_A + R_B + R_C}$$

$$= R_A R_1$$

$$= R_B\, R_2$$

$$= R_C\, R_3$$

Hence,

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

$$= R_2 + R_3 + \frac{R_2 R_3}{R_1}$$

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

$$= R_1 + R_3 + \frac{R_3 R_1}{R_2}$$

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

$$= R_1 + R_2 + \frac{R_1 R_2}{R_3}$$

Thus, delta resistor between the two terminals is the sum of two star resistors connected to the same terminals plus the product of the two resistors divided by the remaining third star resistor.

Note: When three equal resistors are connected in delta, the equivalent star resistance is given by

$$R_Y = \frac{R_\Delta R_\Delta}{R_\Delta + R_\Delta + R_\Delta} = \frac{R_\Delta}{3}$$

or $$R_\Delta = 3R_Y$$

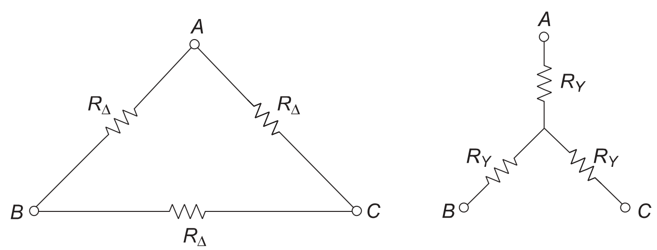


Fig. 2.176

Example 1

Convert the star circuit into its equivalent delta circuit.

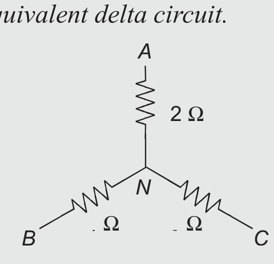


Fig. 2.177 **[May 2015]**

Solution Converting the given star network to delta,

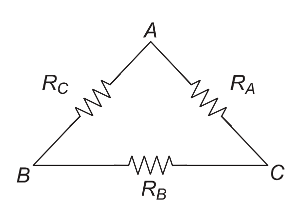


Fig. 2.178

$$R_A = 2 + 6 + \frac{2 \times 6}{4} = 11\,\Omega$$

$$R_B = 6 + 4 + \frac{6 \times 4}{2} = 22\,\Omega$$

$$R_C = 4 + 2 + \frac{4 \times 2}{6} = 7.33\ \Omega$$

Example 2

Find an equivalent resistance between terminals A and B.

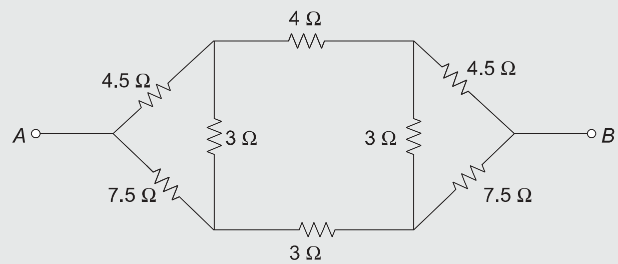


Fig. 2.179

Solution Converting the two delta networks formed by resistors of 4.5 Ω, 3 Ω and 7.5 Ω into equivalent star networks,

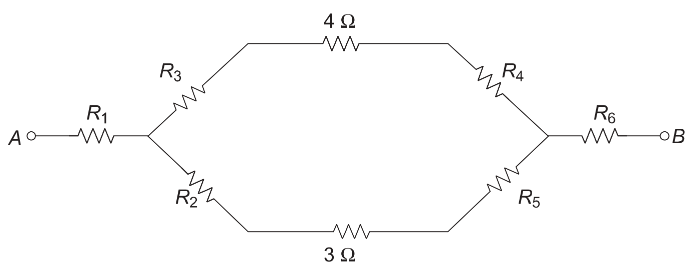


Fig. 2.180

$$R_1 = R_6 = \frac{4.5 \times 7.5}{4.5 + 7.5 + 3} = 2.25\ \Omega$$

$$R_2 = R_5 = \frac{7.5 \times 3}{4.5 + 7.5 + 3} = 1.5\ \Omega$$

$$R_3 = R_4 = \frac{4.5 \times 3}{4.5 + 7.5 + 3} = 0.9\ \Omega$$

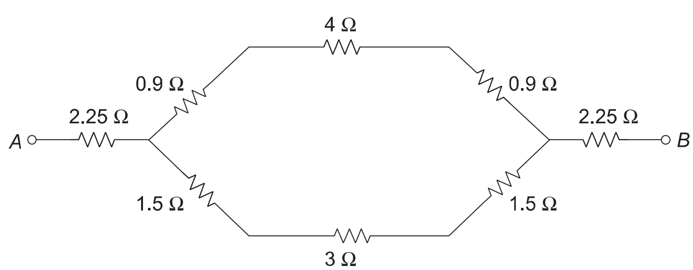


Fig. 2.181

Simplifying the network,

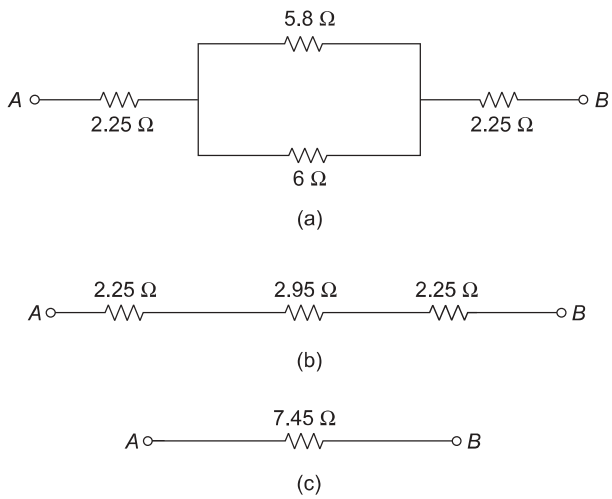


Fig. 2.182

$$R_{AB} = 7.45\ \Omega$$

Example 3

Find an equivalent resistance between terminals A and B.

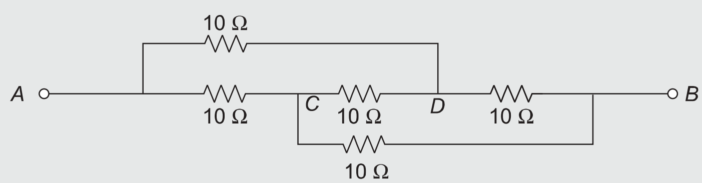


Fig. 2.183

Solution Redrawing the network,

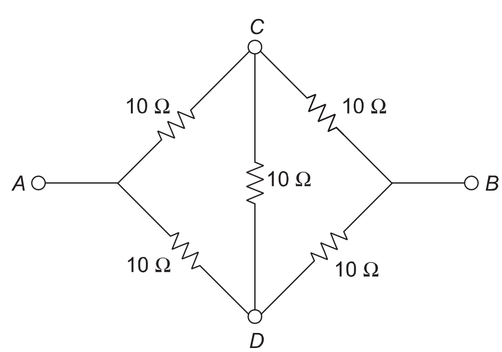


Fig. 2.184

Converting the delta network formed by three resistors of 10 Ω into an equivalent star network,

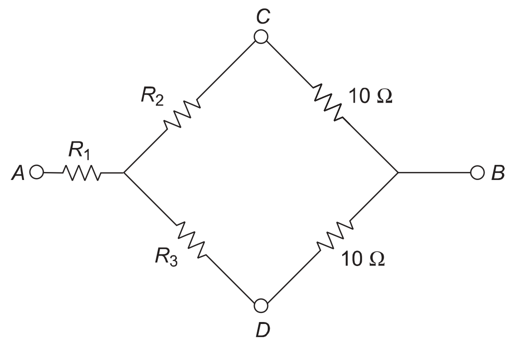


Fig. 2.185

$$R_1 = R_2 = R_3 = \frac{10 \times 10}{10+10+10} = \frac{10}{3}\ \Omega$$

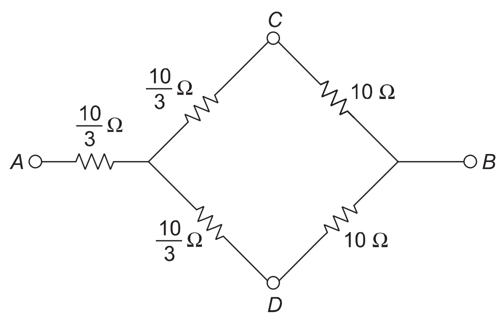


Fig. 2.186

Simplifying the network,

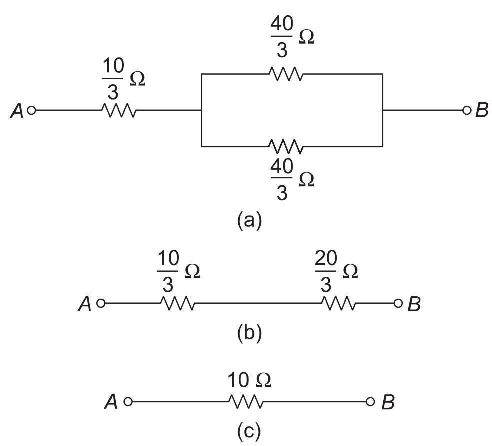


Fig. 2.187

$$R_{AB} = 10\ \Omega$$

Example 4

Calculate R_{xy} for the circuit shown in Fig. 2.188.

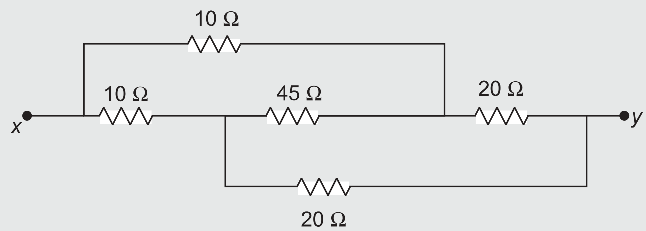


Fig. 2.188 [Dec 2012]

Solution Redrawing the network.

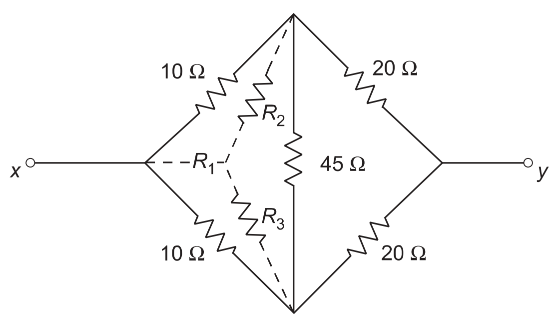


Fig. 2.189

Converting the delta network formed by resistors of 10 Ω, 10 Ω and 45 Ω into an equivalent star network,

$$R_1 = \frac{10 \times 10}{10 + 10 + 45} = 1.54\ \Omega$$

$$R_2 = R_3 = \frac{10 \times 45}{10 + 10 + 45} = 6.92\ \Omega$$

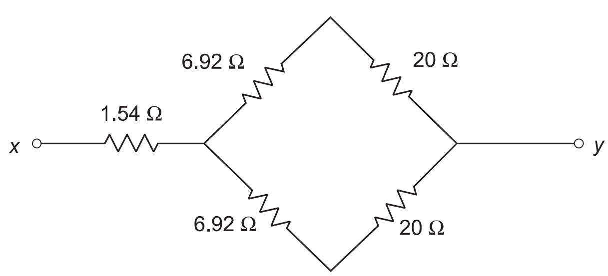


Fig. 2.190

Simplifying the network,

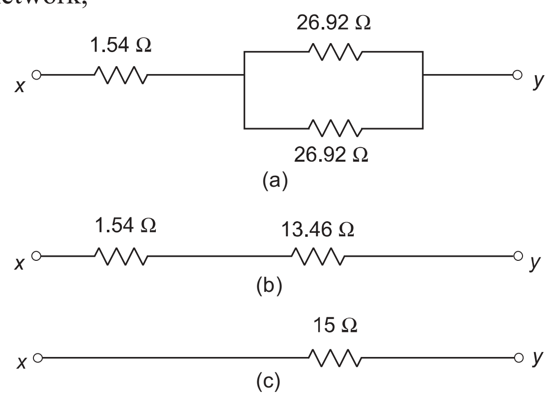


Fig. 2.191

$$R_{xy} = 15\ \Omega$$

Example 5

Find an equivalent resistance between terminals A and B.

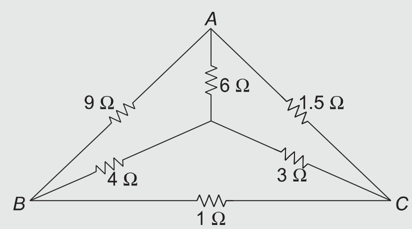


Fig. 2.192

Solution Converting the star network formed by resistors of 3 Ω, 4 Ω and 6 Ω into an equivalent delta network,

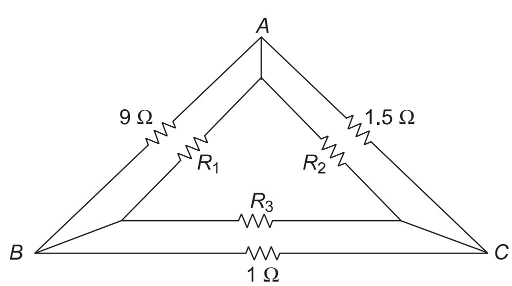


Fig. 2.193

$$R_1 = 6 + 4 + \frac{6 \times 4}{3} = 18\ \Omega$$

$$R_2 = 6 + 3 + \frac{6 \times 3}{4} = 13.5\ \Omega$$

$$R_3 = 4 + 3 + \frac{4\times3}{6} = 9\,\Omega$$

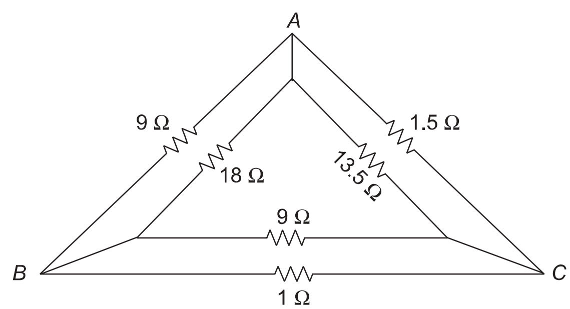


Fig. 2.194

Simplifying the network,

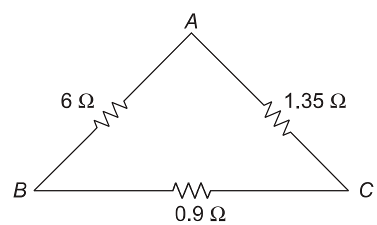


Fig. 2.195

$$\begin{aligned} R_{AB} &= 6 \parallel (1.35 + 0.9) \\ &= 6 \parallel 2.25 \\ &= 1.64\,\Omega \end{aligned}$$

Example 6

Find an equivalent resistance between terminals A and N by solving outer delta ABC.

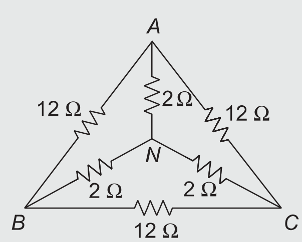


Fig. 2.196

Solution Converting outer delta *ABC* into a star network,

$$R_Y = \frac{12\times12}{12+12+12} = 4\,\Omega$$

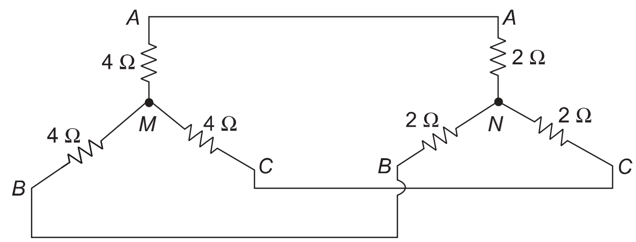


Fig. 2.197

Simplifying the network,

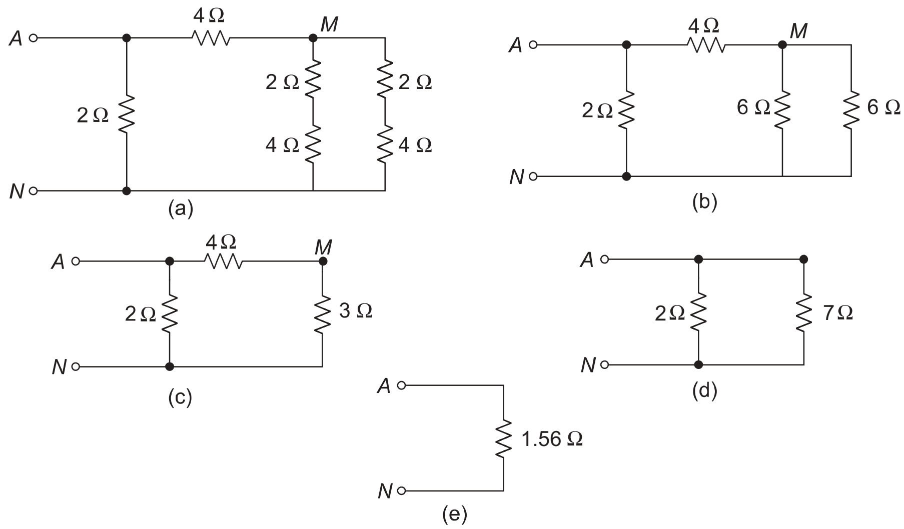


Fig. 2.198

$$R_{AN} = 1.56\ \Omega$$

Example 7

Find an equivalent resistance terminals between A and B.

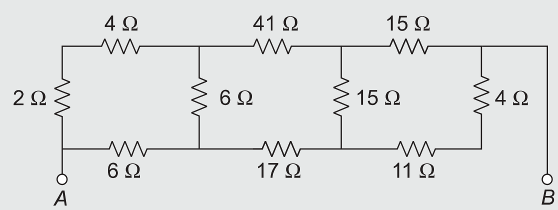


Fig. 2.199

Solution The resistors of 2 Ω and 4 Ω and the resistors of 4 Ω and 11 Ω are connected in series.

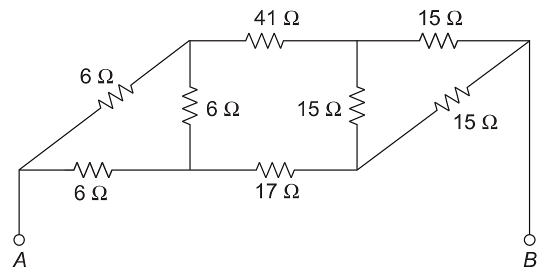


Fig. 2.200

Converting the two outer delta networks into equivalent star networks,

$$R_{Y_1} = \frac{6 \times 6}{6+6+6} = 2\ \Omega$$

$$R_{Y_2} = \frac{15 \times 15}{15+15+15} = 5\ \Omega$$

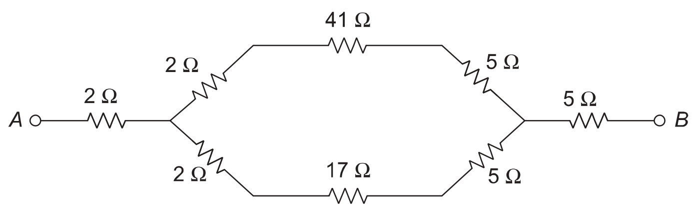


Fig. 2.201

Simplifying the network,

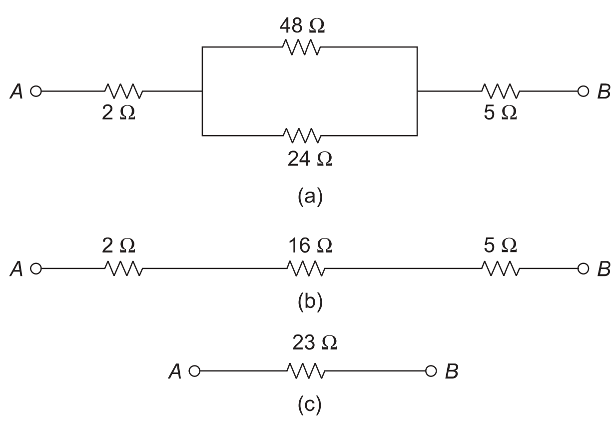


Fig. 2.202

$$R_{AB} = 23\ \Omega$$

Example 8

Find an equivalent resistance between terminals A and B.

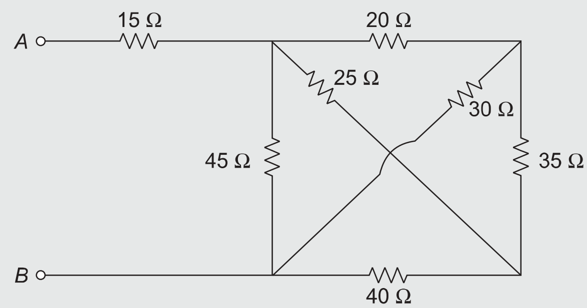


Fig. 2.203 [May 2014]

Solution Drawing the resistor of 30 Ω from outside,

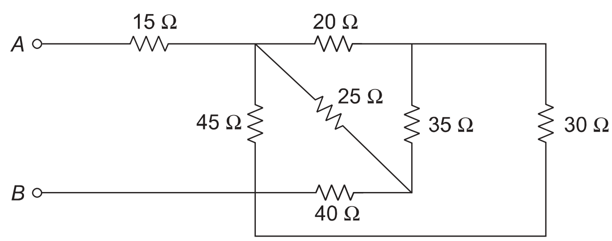


Fig. 2.204

Converting the delta network formed by resistors of 20 Ω, 25 Ω and 35 Ω into an equivalent star network,

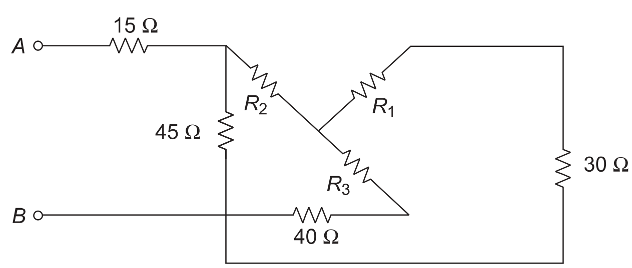


Fig. 2.205

$$R_1 = \frac{20 \times 35}{20+35+25} = 8.75\,\Omega$$

$$R_2 = \frac{20 \times 25}{20+35+25} = 6.25\,\Omega$$

$$R_3 = \frac{35 \times 25}{20+35+25} = 10.94\ \Omega$$

Redrawing the network,

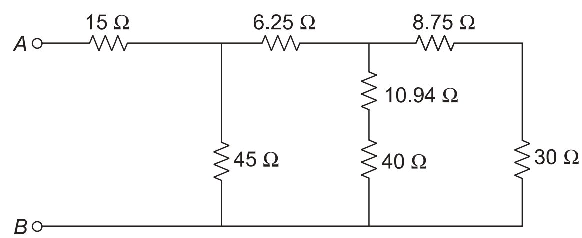


Fig. 2.206

Simplifying the network,

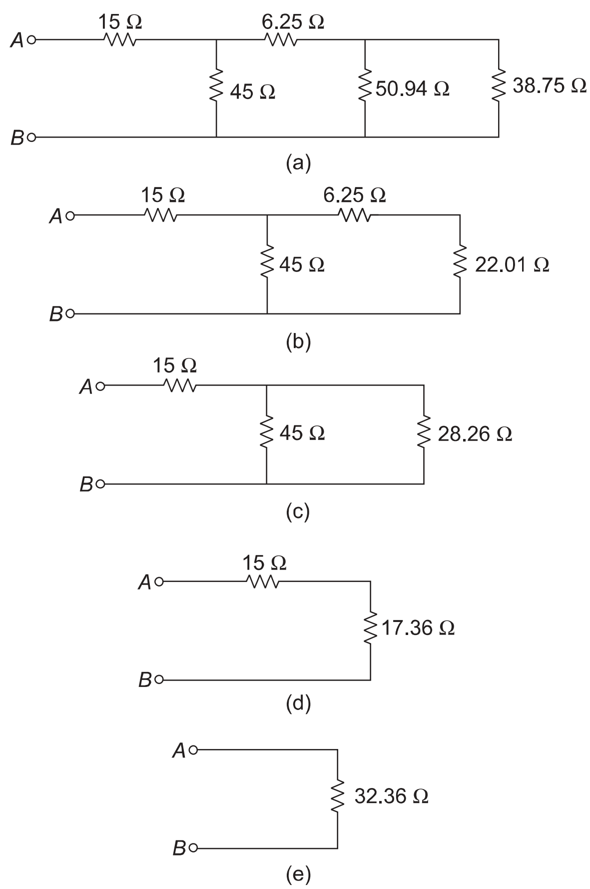


Fig. 2.207

$$R_{AB} = 32.36\ \Omega$$

Example 9

Find an equivalent resistance between terminals A and B.

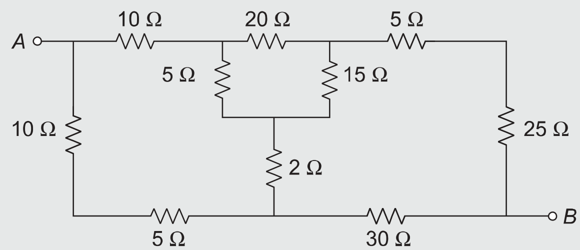


Fig. 2.208

Solution The resistors of 5 Ω and 25 Ω and the resistors of 10 Ω and 5 Ω are connected in series.

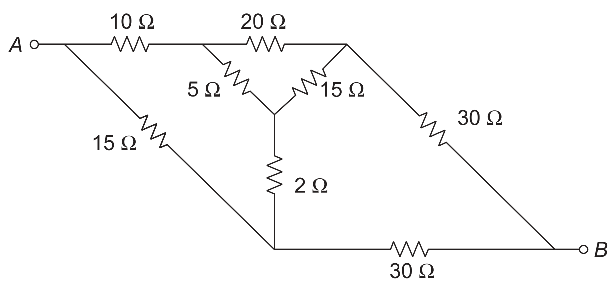


Fig. 2.209

Converting the delta network formed by the resistors of 20 Ω, 5 Ω and 15 Ω into an equivalent star network,

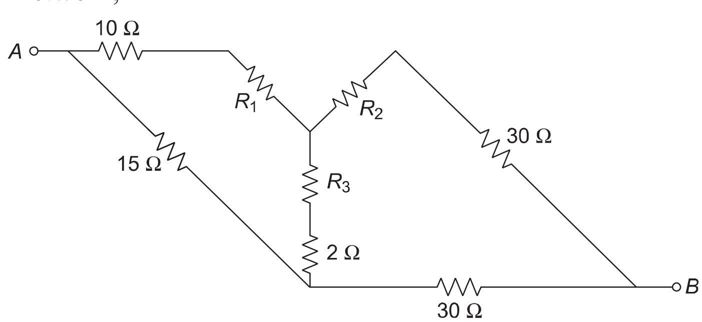


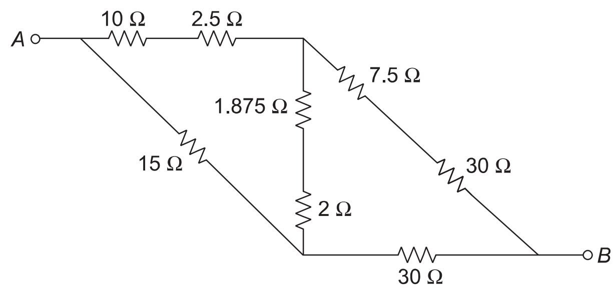
Fig. 2.210

$$R_1 = \frac{20 \times 5}{20+5+15} = 2.5\,\Omega$$

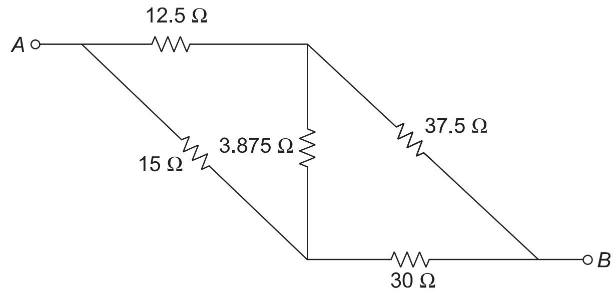
$$R_2 = \frac{20 \times 15}{20+5+15} = 7.5\,\Omega$$

$$R_3 = \frac{5 \times 15}{20+5+15} = 1.875\,\Omega$$

Redrawing the network,



(a)



(b)

Fig. 2.211

Converting the delta network formed by the resistors of 3.875 Ω, 37.5 Ω and 30 Ω into an equivalent star network,

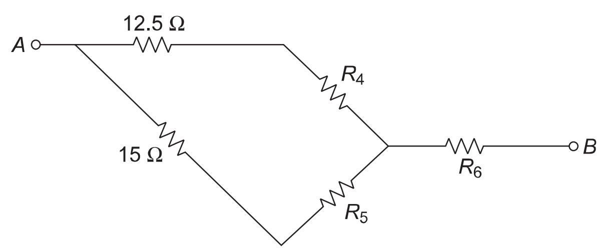


Fig. 2.212

$$R_4 = \frac{3.875 \times 37.5}{3.875 + 37.5 + 30} = 2.04\ \Omega$$

$$R_5 = \frac{3.875 \times 30}{3.875 + 37.5 + 30} = 1.63\ \Omega$$

$$R_6 = \frac{37.5 \times 30}{3.875 + 37.5 + 30} = 15.76\ \Omega$$

Simplifying the network,

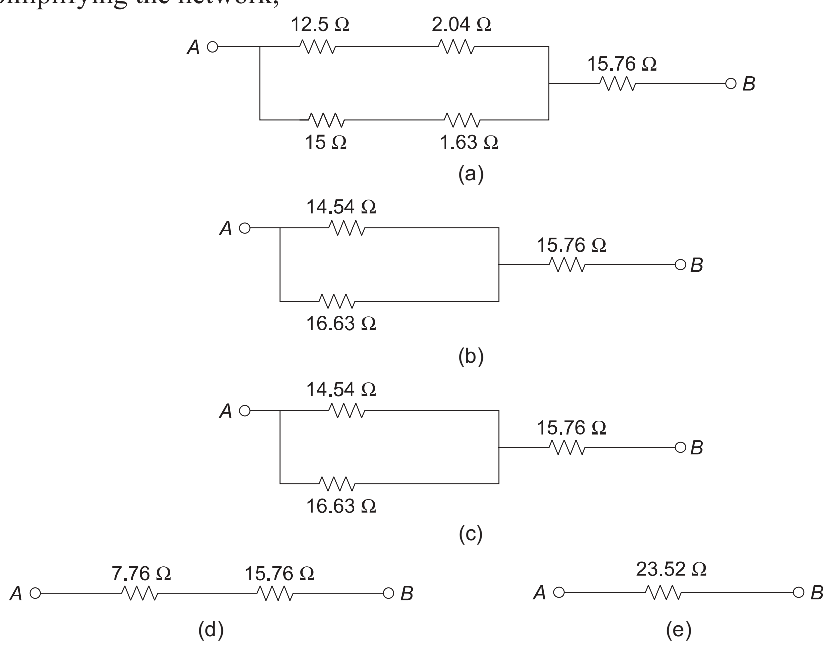


Fig. 2.213

$$R_{AB} = 23.52\ \Omega$$

Example 10

Find an equivalent resistance between terminals A and B.

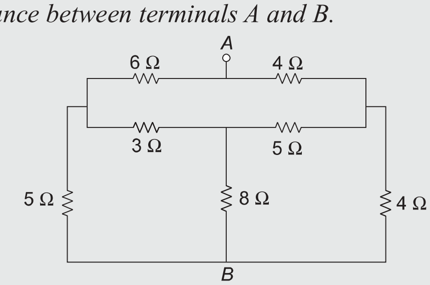


Fig. 2.214

Solution

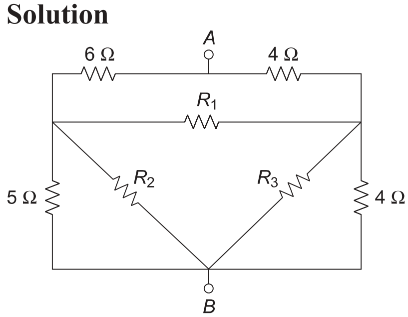


Fig. 2.215

Converting the star network formed by the resistors of 3 Ω, 5 Ω and 8 Ω into an equivalent delta network,

$$R_1 = 3 + 5 + \frac{3\times 5}{8} = 9.875\ \Omega$$

$$R_2 = 3 + 8 + \frac{3\times 8}{5} = 15.8\ \Omega$$

$$R_3 = 5 + 8 + \frac{5\times 8}{3} = 26.33\ \Omega$$

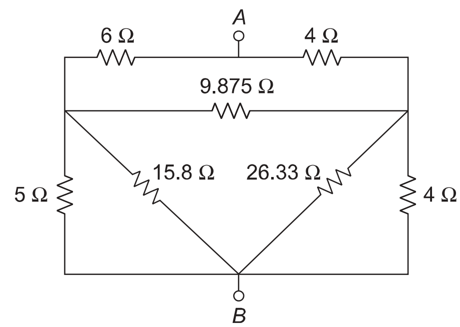


Fig. 2.216

The resistors of 15.8 Ω and 5 Ω and the resistors of 26.33 Ω and 4 Ω are connected in parallel.

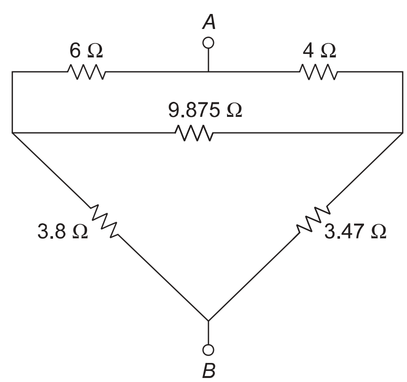


Fig. 2.217

Converting the delta network into a star network,

$$R_4 = \frac{3.8 \times 9.875}{3.8 + 9.875 + 3.47} = 2.19\ \Omega$$

$$R_5 = \frac{3.8 \times 3.47}{3.8 + 9.875 + 3.47} = 0.77\ \Omega$$

$$R_6 = \frac{3.47 \times 9.875}{3.8 + 9.875 + 3.47} = 2\ \Omega$$

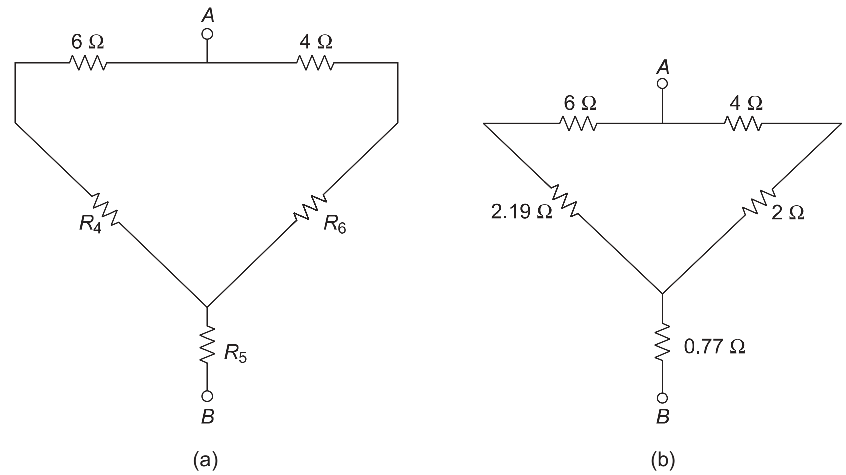


Fig. 2.218

Simplifying the network,

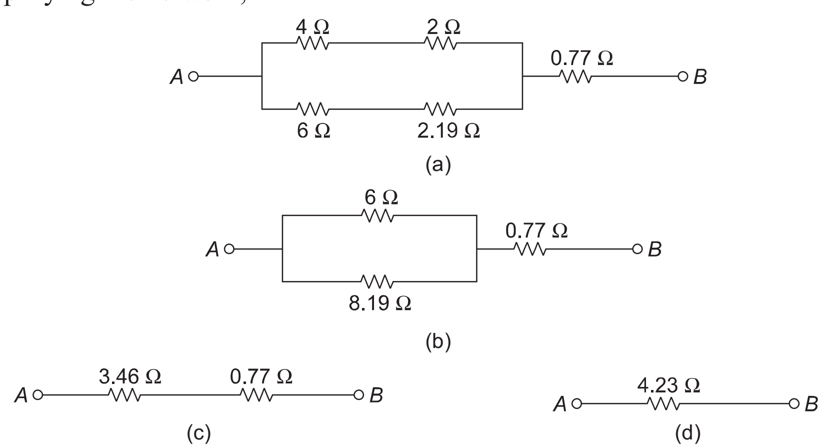


Fig. 2.219

$$R_{AB} = 4.23\ \Omega$$

Example 11

Find an equivalent resistance between terminals A and B.

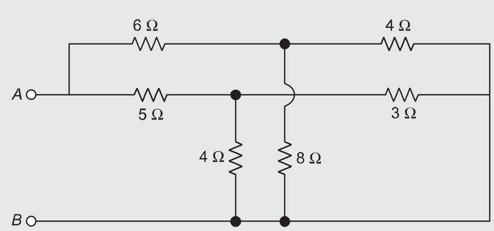


Fig. 2.220

Solution Converting the star network formed by the resistors of 3 Ω, 4 Ω and 5 Ω into an equivalent delta network,

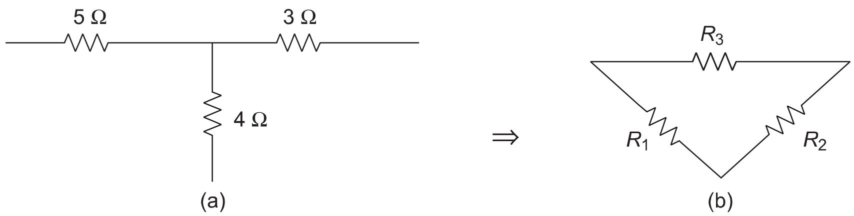


Fig. 2.221

$$R_1 = 5+4+\frac{5\times 4}{3} = 15.67\ \Omega$$

$$R_2 = 3+4+\frac{3\times 4}{5} = 9.4\ \Omega$$

$$R_3 = 5+3+\frac{5\times 3}{4} = 11.75\ \Omega$$

Similarly, converting the star network formed by the resistors of 4 Ω, 6 Ω and 8 Ω into an equivalent delta network,

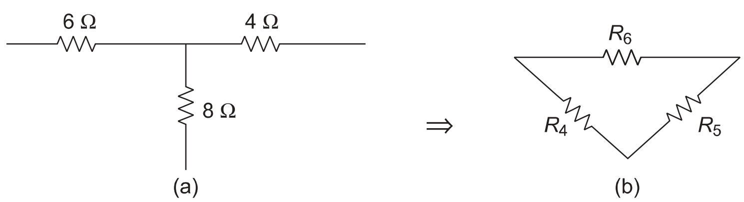


Fig. 2.222

$$R_4 = 6+8+\frac{6\times 8}{4} = 26\ \Omega$$

$$R_5 = 4+8+\frac{4\times 8}{6} = 17.33\ \Omega$$

$$R_6 = 6+4+\frac{6\times 4}{8} = 13\ \Omega$$

These two delta networks are connected in parallel between points *A* and *B*.

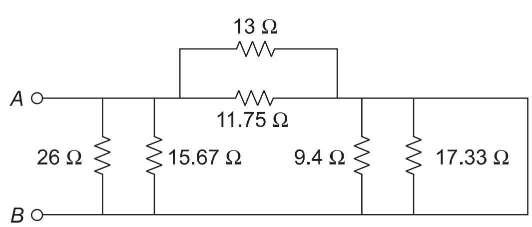


Fig. 2.223

The resistors of 9.4 Ω and 17.33 Ω are in parallel with a short. Hence, the equivalent resistance of this combination becomes zero.

Simplifying the parallel networks,

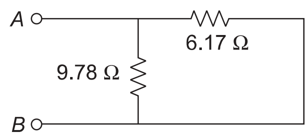


Fig. 2.224

$$R_{AB} = 6.17 \,||\, 9.78 = 3.78\ \Omega$$

Example 12

Find the value of current flowing through 6 Ω resistor.

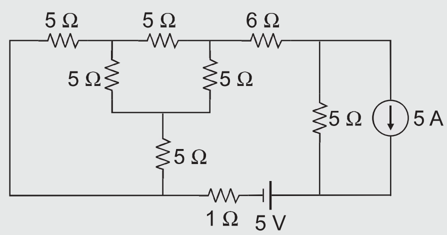


Fig. 2.225 [Dec 2015]

Solution Converting the parallel combination of the current source of 5 A and the resistor of 5 Ω into an equivalent series combination of voltage source and series resistor,

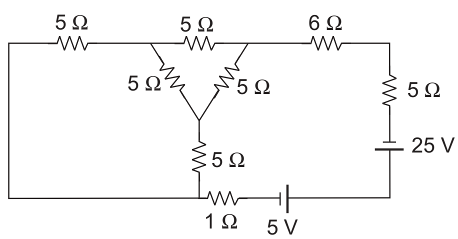


Fig. 2.226

Converting the delta network formed by three 5 Ω resistors into an equivalent star network,

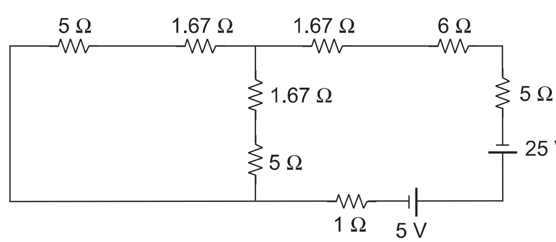


Fig. 2.227

By series-parallel reduction technique and adding two voltage sources,

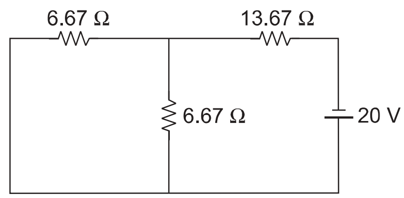


Fig. 2.228

Again by series-parallel reduction technique,

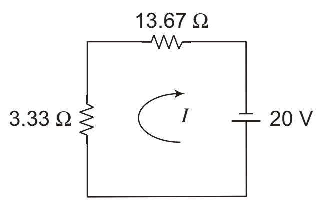


Fig. 2.229

$$I = \frac{20}{13.67 + 3.33} = 1.18 \text{ A}$$

$$I_{6\Omega} = I_{13.67\Omega} = I = 1.18 \text{ A}$$

Example 13

Determine the current supplied by the battery.

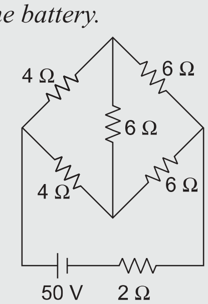


Fig. 2.230

Solution Converting the delta network formed by resistors of 6 Ω, 6 Ω and 6 Ω into an equivalent star network,

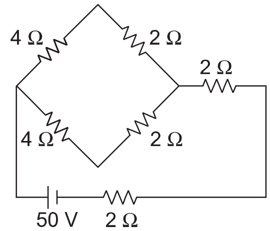


Fig. 2.231

Simplifying the network,

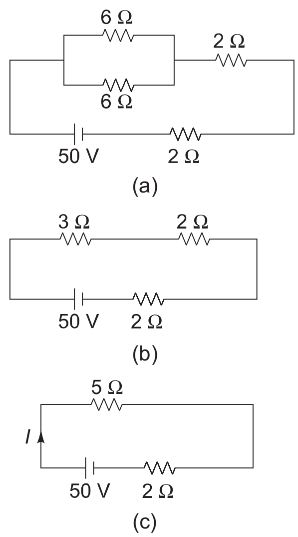


Fig. 2.232

$$I = \frac{50}{5+2} = 7.14 \text{ A}$$

Example 14

Calculate the value of current flowing through the 10 Ω resistor.

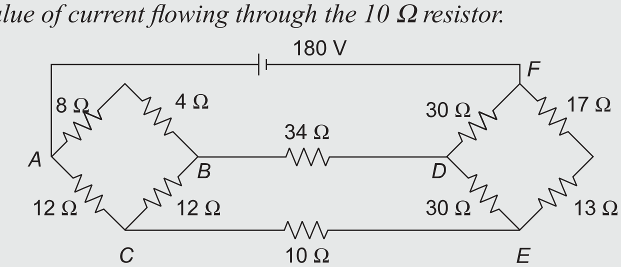


Fig. 2.233

Solution Between terminals *A* and *B* resistors of 8 Ω and 4 Ω are connected in series. Similarly, between terminals *F* and *E*, resistors of 17 Ω and 13 Ω are connected in series.

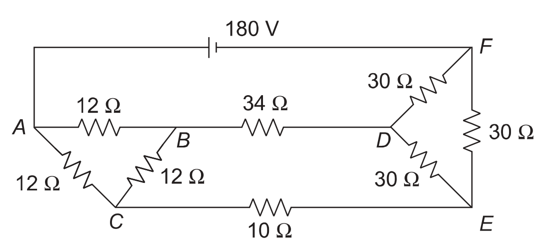


Fig. 2.234

Converting delta *ABC* and *DEF* into an equivalent star network,

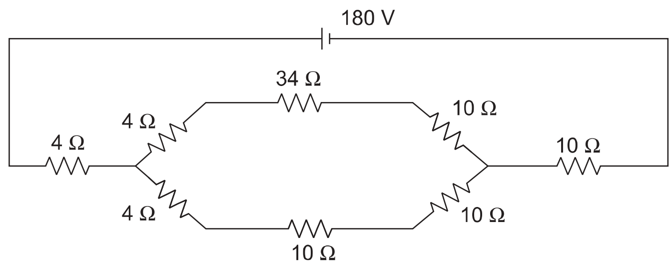


Fig. 2.235

Simplifying the network,

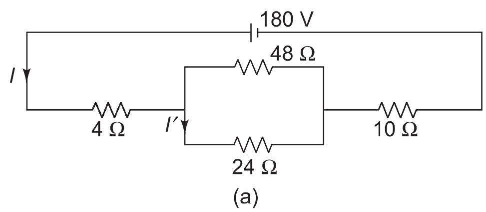


Fig. 2.236

$$I = \frac{180}{4+16+10} = 6\text{ A}$$

By current-division rule,

$$I' = I_{24\,\Omega} = I_{10\,\Omega} = 6 \times \frac{48}{24+48} = 4\text{ A}$$

Example 15

Determine current flow through the 20 Ω resistor in the following circuit in Fig. 2.237.

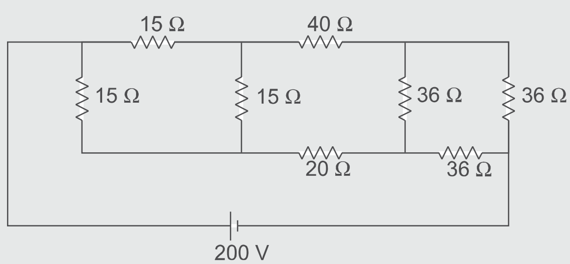


Fig. 2.237 [Dec 2013]

Solution Converting the two outer delta networks into equivalent star networks,

$$R_{Y1} = \frac{15 \times 15}{15+15+15} = 5\ \Omega$$

$$R_{Y2} = \frac{36 \times 36}{36+36+36} = 12\ \Omega$$

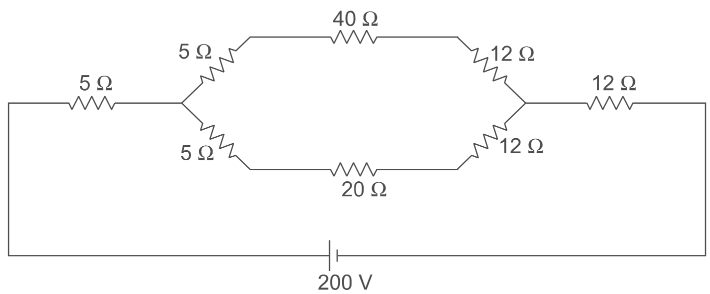


Fig. 2.238

Simplifying the network,

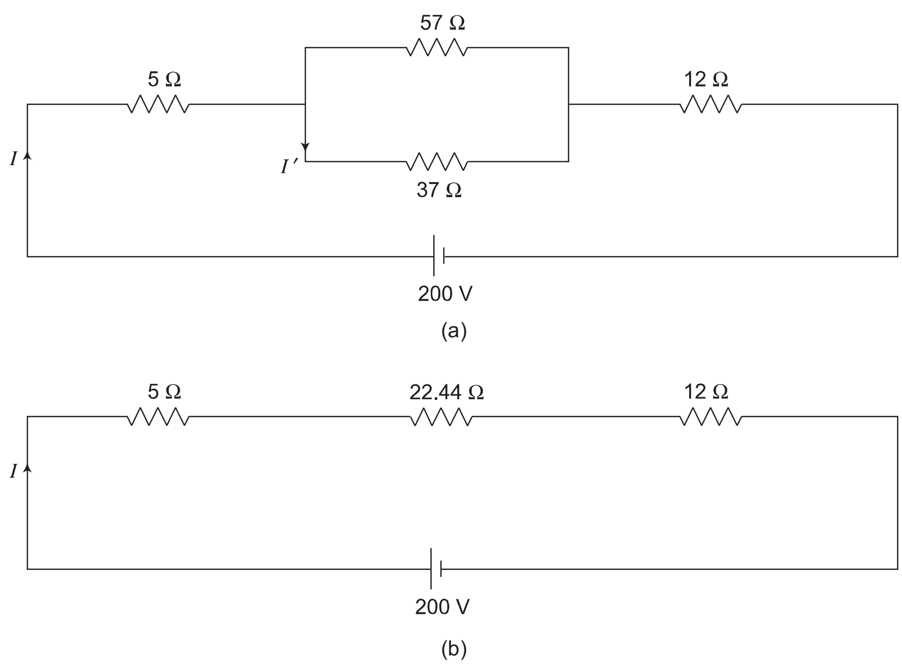


Fig. 2.239

$$I = \frac{200}{5 + 22.44 + 12} = 5.07\ \text{A}$$

By current-division rule,

$$I_{20\,\Omega} = I_{37\,\Omega} = 5.07 \times \frac{57}{57 + 37} = 3.07\ \text{A}$$

Example 16

Find the current supplied by the battery.

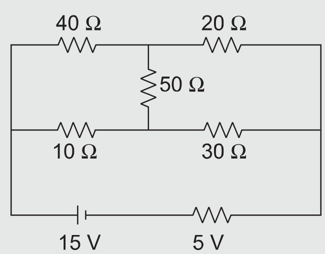


Fig. 2.240

Solution Converting the star network formed by resistors of 40 Ω, 20 Ω and 50 Ω into an equivalent delta network,

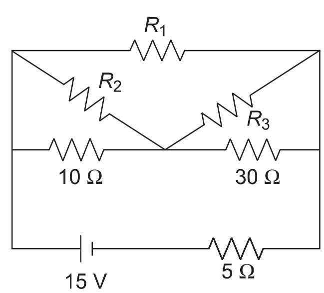


Fig. 2.241

$$R_1 = 40 + 20 + \frac{40 \times 20}{50} = 76\ \Omega$$

$$R_2 = 40 + 50 + \frac{40 \times 50}{20} = 190\ \Omega$$

$$R_3 = 20 + 50 + \frac{20 \times 50}{40} = 95\ \Omega$$

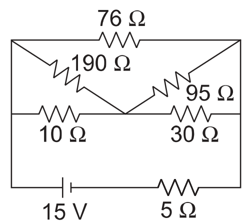


Fig. 2.242

The resistors of 190 Ω and 10 Ω and the resistors of 95 Ω and 30 Ω are connected in parallel.

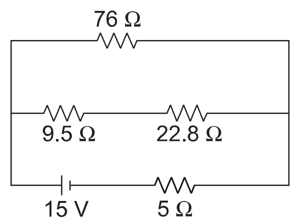


Fig. 2.243

Simplifying the network,

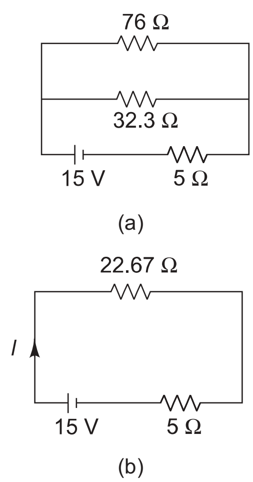
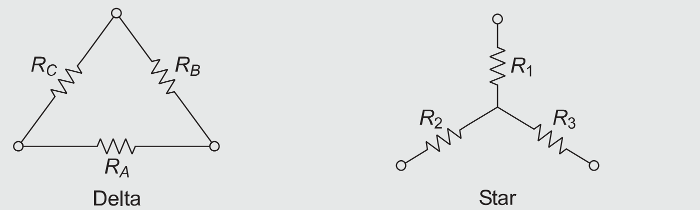


Fig. 2.244

$$I = \frac{15}{22.67 + 5} = 0.542 \text{ A}$$

Useful Formulae



Delta to star transformation

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C}$$

$$R_2 = \frac{R_A R_C}{R_A + R_B + R_C}$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C}$$

Star to delta transformation

$$R_A = R_2 + R_3 + \frac{R_2 R_3}{R_1}$$

$$R_B = R_1 + R_3 + \frac{R_1 R_3}{R_2}$$

$$R_C = R_1 + R_2 + \frac{R_1 R_2}{R_3}$$

Exercise 2.5

2.1 Find the equivalent resistance between terminals *A* and *B*.

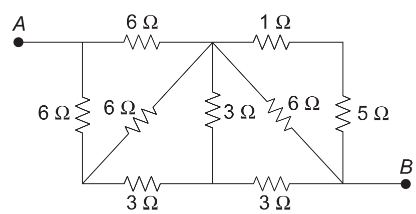


Fig. 2.245

[5 Ω]

2.2 Find the equivalent resistance between terminals *A* and *B*.

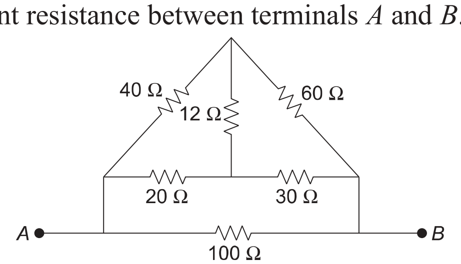


Fig. 2.246

[25 Ω]